Coalition games over Kripke semantics: expressiveness and complexity

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ABSTRACT. We show that Pauly's Coalition Logic can be embedded into a richer normal modal logic that we call Normal Simulation of Coalition Logic (NCL). We establish that the latter is strictly more expressive than the former by proving that it is NEXPTIME complete in the case of at least two agents.

1 Introduction

Recently Shahid Rahman, together with Cédric Dégremont [Dég06], has proposed a dialogical proof system for the deliberative STIT theories [BPX01].

At the same time, two of us were investigating the properties of these theories, as well as its relationship with Coalition Logic, Alternating-time Temporal Logic (ATL) and epistemic extensions of these [BHT06a, HT06, BHT07]. (See also [Tro07].) The work of Cédric Dégremont shed new light on a subject that was studied only very rarely up to now viz. the proof-theory of deliberative STIT-theories (a notable exception being [Wan06]). Their contribution was one of the triggers of our interest in proof systems for logics of agency, paving the way for decidability and complexity results. The present paper provides such results for a fragment of the logic of the Chellas STIT, viz. Coalition Logic.

Coalition Logic (CL) was proposed by Pauly in [Pau01] as a logic for reasoning about social procedures characterized by complex strategic interactions between agents, individuals or groups. Examples of such procedures are fair-division algorithms or voting processes. CL facilitates reasoning about abilities of coalitions in games by extending classical logic with operators $\langle\!\langle J \rangle\!\rangle\varphi$ for groups of agents J, reading: "the coalition J has a joint strategy to ensure that φ".[1]

[1] Note that we use $\langle\!\langle J \rangle\!\rangle\varphi$ as an alternative notation for Pauly's non-normal operator $[J]\varphi$. We introduce this alternative syntax for two reasons: (1) the new syntax fits better with the quantifier combination $\exists - \forall$ underlying the semantics, and (2) we use Pauly's

In [BHT06b], we have shown that CL can be embedded into the logic of the Chellas STIT. STIT theory is the most prominent account of agency in philosophy of action. It is the logic of constructions of the form "agent i sees to it that φ holds". In the present paper we go beyond that and provide a proof-theoretic analysis of the embedded fragment. In order to do that, we extend Xu's logic of the Chellas's STIT-logic with a 'next' operator, resulting in a logic we call NCL. We provide a complete and elegant axiomatization and prove that Xu's logic of the Chellas's STIT-logic and CL are embedded.

As designed by Pauly, semantics of Coalition Logic is in terms of neighborhood models, that is, models providing a *neighborhood function*, associating a world to a set of neighborhoods, or clusters. (See [Che80, Chap. 7] for details about those models.) Here, we present the normal logic NCL whose semantics is the well known *relational* or *Kripke semantics*. Our embedding is in itself an interesting result since it shows that Coalition Logic can be evaluated with respect to relational models.

Moreover, NCL extends CL with capabilities of reasoning about what a coalition is actually doing or about to do, as opposed to what it *could* do. We finally establish decidability and complexity results.

The remaining of this paper is along the following outline. Section 2 and 3 present respectively Coalition Logic and its normal simulation NCL. In Section 4, we check that NCL inherits all the principles of a version of the deliberative STIT theories without tense operators. Analogously, Section 5 provides a translation from Coalition Logic to NCL. Section 6 is devoted to the studies of decidability and complexity. In Section 7, we devise about NCL expressiveness. In particular, we enlighten that NCL is more expressive than Coalition Logic, and informally discuss a possible application of the logic to the notion of 'power over' of agents and group of agents. We finally conclude in Section 8.

2 Coalition Logic CL

2.1 Syntax of CL

Let AGT be a nonempty finite set of agents and $Prop$ an infinite countable set of atomic formulas. The language $\mathcal{L}_{CL}$ (all formulas of Coalition Logic) is defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\!\langle J \rangle\!\rangle\varphi$$

where p ranges over $Prop$ and J ranges over subsets of AGT. The other Boolean operators are defined as usual.

original syntax $[J]\varphi$ to denote Chellas's STIT operator, thereby emphasizing that this is a normal modal necessity operator.

2.2 Coalition model semantics

DEFINITION 1 (effectivity function). Given a nonempty set of states S, an *effectivity function* is a function $E : 2^{AGT} \to 2^{2^S}$. An effectivity function is said to be:

- *J-maximal* iff for all $X \subseteq S$, if $S \setminus X \notin E(AGT \setminus J)$ then $X \in E(J)$;
- *outcome monotonic* iff for all $X, X' \subseteq S$ and for all $J \subseteq AGT$, if $X \in E(J)$ and $X \subseteq X'$ then $X' \in E(J)$;
- *superadditive* iff for all J_1, J_2, if $J_1 \cap J_2 = \emptyset$ then for all $X_1, X_2 \subseteq S$, if $X_1 \in E(J_1)$ and $X_2 \in E(J_2)$ then $X_1 \cap X_2 \in E(J_1 \cup J_2)$.

The function E intuitively associates every coalition J to a set of subsets of S (or set of outcomes) for which J is effective. That is, J can force the world to be in some state of X, for each $X \in E(J)$.

DEFINITION 2 (playable effectivity function). Given a nonempty set of states S, an effectivity function $E : 2^{AGT} \to 2^{2^S}$ is said to be *playable* iff the following conditions hold:

1. for all J, $\emptyset \notin E(J)$ (Liveness)
2. for all J, $S \in E(J)$ (Termination)
3. E is AGT-maximal
4. E is outcome-monotonic
5. E is superadditive

A coalition model is a pair $((S, E), V)$ where:

- S is a nonempty set of states;
- $E : S \to (2^{AGT} \to 2^{2^S})$ associates every state s with a playable effectivity function $E(s)$.
- $V : S \to 2^{Prop}$ is a valuation function.

We will write $E_s(J)$ instead of $E(s)(J)$ to denote the effectivity of J at the state s.

Truth conditions are standard for Boolean operators. We evaluate the coalitional operators against a coalition model M and a state s as follows:

$$M, s \models \langle\!\langle J \rangle\!\rangle \varphi \text{ iff } \{t \mid M, t \models \varphi\} \in E_s(J)$$

2.3 Game semantics

In [Pau02], Pauly investigates the link between coalition models and strategic games.

DEFINITION 3. A *strategic game* is a tuple $G = (S, \{\Sigma_i | i \in AGT\}, o)$ where S is a nonempty set, Σ_i is a nonempty set of choices for every agent $i \in AGT$, $o : \prod_{i \in AGT} \Sigma_i \rightarrow S$ is an outcome function which associates an outcome state in S with every combination of choice of agents (choice profile).

It appears that there is a strong link between a coalition model (whose effectivity structure is *playable* by definition) and a strategic game.

DEFINITION 4. Given a strategic game $G = (S, \{\Sigma_i | i \in AGT\}, o)$, the *effectivity function* $E_G : 2^{AGT} \rightarrow 2^{2^S}$ of G is defined as follows: for all J, let $E_G(J)$ be the set of all subsets X of S such that there exists a $Card(J)$-tuple σ in $\prod_{i \in J} \Sigma_i$ such that for all $Card(AGT \setminus J)$-tuples σ' in $\prod_{i \in AGT \setminus J} \Sigma_i$, $o(\sigma, \sigma')$ is in X.

Pauly then gives the following characterization:

THEOREM 5 ([Pau02]). *An effectivity function E is playable iff it is the effectivity function of some strategic game.*

DEFINITION 6. Let $((S, E), V)$ be a coalition model. Let s be a state of S. A set $Y \subseteq S$ is called a *minimal effectivity outcome at s for J* iff (1) $Y \in E_s(J)$ and (2) for all $Y' \in E_s(J)$, if $Y' \subseteq Y$ then $Y' = Y$.

DEFINITION 7. The *non-monotonic core* of E is the mapping $\mu_E : 2^{AGT} \times S \rightarrow 2^{2^S}$ such that $\mu_E(J, s) = \{Y \mid Y$ is a minimal effectivity outcome at s for $J\}$.

The outcome of a strategic game is completely determined when every agent has made its choice.

PROPOSITION 8. *Let (S, E, V) be a coalition model. For all states $s \in S$, $\mu_E(AGT, s)$ is a nonempty set of singletons.*

PROOF. This is a corollary of Theorem 5. ■

2.4 Axiomatization of CL

The set of formulas that are valid in coalition models is completely axiomatized by the following principles [Pau02].

(ProTau) enough tautologies of propositional calculus

($\perp$) $\neg \langle\!\langle J \rangle\!\rangle \perp$

$(\top)$	$\langle\!\langle J \rangle\!\rangle \top$
(N)	$\neg \langle\!\langle \emptyset \rangle\!\rangle \neg\varphi \rightarrow \langle\!\langle AGT \rangle\!\rangle \varphi$
(M)	$\langle\!\langle J \rangle\!\rangle (\varphi \wedge \psi) \rightarrow \langle\!\langle J \rangle\!\rangle \varphi \wedge \langle\!\langle J \rangle\!\rangle \psi$
(S)	$\langle\!\langle J_1 \rangle\!\rangle \varphi \wedge \langle\!\langle J_2 \rangle\!\rangle \psi \rightarrow \langle\!\langle J_1 \cup J_2 \rangle\!\rangle (\varphi \wedge \psi)$ if $J_1 \cap J_2 = \emptyset$
(MP)	from φ and from $\varphi \rightarrow \psi$ infer ψ
(RE)	from $\varphi \leftrightarrow \psi$ infer $\langle\!\langle J \rangle\!\rangle \varphi \leftrightarrow \langle\!\langle J \rangle\!\rangle \psi$

Note that the (N) axiom corresponds to the determinism of *choice profiles* (actions constituted by concurrent choices for every agent in the system): when every agent opts for a choice, the next state is fully determined, thus, if a formula is not settled true, the coalition of all agents (AGT) can always work together to make its negation true. The axiom (S) says that two disjoint coalitions can combine their efforts to ensure a conjunction of properties. Note that from (RE), (S) and $(\bot)$ it follows that if J_1 and J_2 are disjoint then $\langle\!\langle J_1 \rangle\!\rangle \varphi \wedge \langle\!\langle J_2 \rangle\!\rangle \neg\varphi$ is inconsistent. So, two disjoint coalitions cannot consistently ensure opposed facts.

3 Normal simulation of Coalition Logic NCL

3.1 Syntax of NCL

Let AGT be a nonempty finite set of agents and $Prop$ an infinite countable set of atomic formulas. Without loss of generality, we assume that $AGT = \{0, \ldots, n-1\}$ where $n = Card(AGT)$. The language $\mathcal{L}_{NCL}$ of NCL has the following syntax, where p ranges over elements of $Prop$ and J ranges over subsets of AGT:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid [J]\varphi$$

The other Boolean operators are obtained as usual, and $\langle J \rangle \varphi =_{def} \neg[J]\neg\varphi$.

3.2 Semantics of NCL

The models of NCL are tuples $\mathcal{M} = (W, R, F_X, \pi)$ where:

- W is a nonempty set of worlds (alias contexts);
- R is a collection of equivalence relations R_J (one for every coalition $J \subseteq AGT$) such that:
  - $R_{J_1 \cup J_2} \subseteq R_{J_1} \cap R_{J_2}$
  - $R_\emptyset \subseteq R_J \circ R_{AGT \setminus J}$
  - $R_{AGT} = Id$

- $F_X : W \to W$ is a total function;
- $\pi : W \to 2^{Prop}$ is a valuation function.

The truth conditions of the operators are given by:

- $\mathcal{M}, w \models \mathbf{X}\varphi$ iff $\mathcal{M}, F_X(w) \models \varphi$
- $\mathcal{M}, w \models [J]\varphi$ iff $\forall u \in R_J(w),\ \mathcal{M}, u \models \varphi$

3.3 Axiomatization of NCL

We give the following axiom schemas for NCL.

(ProTau)	enough tautologies of propositional calculus
S5($[J]$)	all S5-theorems, for every $[J]$
(Mon)	$[J_1]\varphi \vee [J_2]\varphi \to [J_1 \cup J_2]\varphi$
Elim($\emptyset$)	$\langle\emptyset\rangle\varphi \to \langle J\rangle\langle AGT \setminus J\rangle\varphi$
Triv(AGT)	$\varphi \to [AGT]\varphi$
K($\mathbf{X}$)	all K-theorems for $\mathbf{X}$
D($\mathbf{X}$)	$\mathbf{X}\varphi \to \neg\mathbf{X}\neg\varphi$
Det(X)	$\neg\mathbf{X}\neg\varphi \to \mathbf{X}\varphi$

We admit the standard inference rules of modus ponens and necessitation for $[\emptyset]$ and $\mathbf{X}$. From the former, necessitation for every $[J]$ follows by the inclusion axiom (Mon). A formula φ is a theorem of NCL, in symbols $\vdash_{\mathsf{NCL}} \varphi$, iff it can be derived from the above axioms and inference rules within a finite number of steps.

LEMMA 9. $\vdash_{\mathsf{NCL}} \langle\emptyset\rangle\varphi \to \langle J_1\rangle\langle J_2\rangle\varphi$ *if* $J_1 \cap J_2 = \emptyset$.

PROOF. By Elim($\emptyset$) we have $\vdash_{\mathsf{NCL}} \langle\emptyset\rangle\varphi \to \langle J_1\rangle\langle AGT \setminus J_1\rangle\varphi$. Now by hypothesis $J_1 \cap J_2 = \emptyset$, or equivalently $J_2 \subseteq AGT \setminus J_1$. Thus by (Mon) $\vdash_{\mathsf{NCL}} \langle AGT \setminus J_1\rangle\varphi \to \langle J_2\rangle\varphi$. We obtain $\vdash_{\mathsf{NCL}} \langle J_1\rangle\langle AGT \setminus J_1\rangle\varphi \to \langle J_1\rangle\langle J_2\rangle\varphi$ by $[J_1]$-necessitation and K($[J_1]$). We conlude that $\vdash_{\mathsf{NCL}} \langle\emptyset\rangle\varphi \to \langle J_1\rangle\langle J_2\rangle\varphi$. ■

THEOREM 10. *Our axiomatization of NCL is both sound and complete with respect to the class of all models of NCL, i.e.,* $\vdash_{\mathsf{NCL}} \varphi$ *iff for all models* $\mathcal{M} = (W, R, F_X, \pi)$ *of NCL and for all worlds* $w \in W$, $\mathcal{M}, w \models \varphi$.

PROOF. Soundness is obtained by a routine argument while completeness is immediate from Sahlqvist's theorem. Cf. [BdRV01]. ■

4 Translating Chellas's STIT logic into NCL

We call here Chellas's STIT logic a possible presentation of the so-called *deliberative STIT theories* [HB95] without tense operators. An axiomatics was provided by Xu and named *Ldm* in [BPX01, Chap. 17]. It is the logic of the Chellas's STIT operators for individual agents plus an operator for historical necessity $\Box$. Recently in [BHT08], three of us have shown that *Ldm* axiomatics could be simplified and in particular that historical necessity was superfluous in presence of at least two agents. This work resulted in an alternative axiomatics of *Ldm* noted *ALdm*.

Via *ALdm*, we show that Chellas's STIT logic embeds in NCL.

4.1 Syntax of *ALdm*

Let AGT be a nonempty finite set of agents and $Prop$ an infinite countable set of atomic formulas. Without loss of generality, we assume that $AGT = \{0, \ldots, k\}$ where k is a non-negative integer. The language $\mathcal{L}_{ALdm}$ of $ALdm$ has the following syntax, where p ranges over elements of $Prop$ and i ranges elements of AGT:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid [i]\varphi \mid \Box\varphi$$

The other classical connectives are obtained as usual. Let $\langle i\rangle\varphi =_{def} \neg[i]\neg\varphi$ and $\Diamond\varphi =_{def} \neg\Box\neg\varphi$.

4.2 Semantics for *ALdm*

DEFINITION 11. An *ALdm*-model is a tuple $M = (W, R, V)$ where:

- W is a nonempty set of contexts;
- R is a collection of equivalence relations R_i (one for every agent $i \in AGT$) such that for all $w, v \in W$ and for all $l, m, n \in AGT$, if $(w, v) \in R_l \circ R_m$, then $\exists u \in W, (w, u) \in R_n$ and $(u, v) \in \bigcap_{i \in AGT \setminus \{n\}} R_i$;
- $V : W \to 2^{Prop}$ is a valuation function.

Truth conditions are as follows:

- $M, w \models [i]\varphi$ iff for all $u \in R_i(w)$, $\mathcal{M}, u \models \varphi$
- $M, w \models \Box\varphi$ iff for all $u \in (R_1 \circ R_0)(w)$, $\mathcal{M}, u \models \varphi$

and as usual for Boolean operators.

4.3 *ALdm* axiomatics

Axiom schemas for *ALdm* are given as follows:

S5(i)	all S5-theorems, for every $[i]$
Def($\Box$)	$\Box\varphi \leftrightarrow [1][0]\varphi$
(GPerm$_k$)	$\langle l\rangle\langle m\rangle\varphi \to \langle n\rangle \bigwedge_{i\in AGT\setminus\{n\}}\langle i\rangle\varphi$

In the axiom (GPerm$_k$), note that $l, m, n \in AGT$. As proved in [BHT08], from (GPerm$_k$) we can derive Xu's "axiom scheme for independence of agents" (AIA$_k$) [BPX01, Chap. 17]. For instance, (AIA$_1$) corresponds to:[2]

$$\Diamond[0]\varphi_0 \wedge \Diamond[1]\varphi_1 \to \Diamond([0]\varphi_0 \wedge [1]\varphi_1)$$

(AIA$_k$) is central in STIT since it captures the notion of independence of agents. We show later in Lemma 15 that a group version of (AIA$_1$) is a theorem of NCL too.

THEOREM 12 ([BHT08]). *Our axiomatization of ALdm is both sound and complete with respect to the class of all ALdm-models.*

4.4 Embedding *ALdm* in NCL

NCL is easily proved to be a conservative extension of *ALdm*. To illustrate that, we give the following translation from formulas of *ALdm* to formulas of NCL.

$$\begin{array}{lcl} tr_0(p) & = & p \\ tr_0(\Box\varphi) & = & [\emptyset]\varphi \\ tr_0([i]\varphi) & = & [\{i\}]tr_0(\varphi) \end{array}$$

and homomorphic for the Boolean operators.

LEMMA 13. *The translation of (GPerm$_k$) by tr_0 is a theorem of NCL.*

PROOF. By applying (Mon) to $\langle\{l\}\rangle$ and $\langle\{m\}\rangle$ in the right part of the tautology $\langle\{l\}\rangle\langle\{m\}\rangle\varphi \to \langle\{l\}\rangle\langle\{m\}\rangle\varphi$ and next S5($[\emptyset]$) we have $\vdash_{\mathsf{NCL}} \langle\{l\}\rangle\langle\{m\}\rangle\varphi \to \langle\emptyset\rangle\varphi$. Then by Elim($\emptyset$) we obtain $\vdash_{\mathsf{NCL}} \langle\{l\}\rangle\langle\{m\}\rangle\varphi \to \langle\{n\}\rangle\langle AGT \setminus \{n\}\rangle\varphi$.

Now, by classical principles on instances of (Mon) $\langle AGT \setminus \{n\}\rangle\varphi \to \langle\{i\}\rangle\varphi$ for every $i \in AGT \setminus \{n\}$, we have $\vdash_{\mathsf{NCL}} \langle AGT \setminus \{n\}\rangle\varphi \to \bigwedge_{i\in AGT\setminus\{n\}}\langle\{i\}\rangle\varphi$. We conclude that $\vdash_{\mathsf{NCL}} \langle\{l\}\rangle\langle\{m\}\rangle\varphi \to \langle\{n\}\rangle \bigwedge_{i\in AGT\setminus\{m\}}\langle\{i\}\rangle\varphi$. ■

We prove that NCL is a conservative extension of *ALdm* in presence of at least two agents.

[2] (AIA$_1$) corresponds to the case of two agents 0 and 1.

THEOREM 14. *φ is a theorem of $ALdm$ iff $tr_0(\varphi)$ is a theorem of NCL.*

PROOF.
$\boxed{\Rightarrow}$ Remind that besides (GPerm_k), the only other axioms of $ALdm$ are $S5$ axioms for $[i]$. From S5($[J]$) and Lemma 13, we have that every translated axiom of $ALdm$ is a theorem of NCL. Moreover, translated inference rules preserve validity.
$\boxed{\Leftarrow}$ Let $M = \langle W, R, V\rangle$ be an $ALdm$-model, $x \in W$ be a world and φ a $ALdm$-formula.
Assume $M, x \models \varphi$. We transform M into an NCL-model $\mathcal{M} = (W', R', F'_X, \pi)$ as follows:

- $W' = W$;
- $R'_\emptyset = R_1 \circ R_0$;
- $R'_{AGT} = id$;
- $R'_J = \bigcap_{j \in J} R_j$, if $J \neq \emptyset$ and $J \neq AGT$;
- $F'_X = id$;
- $\pi(w) = V(w)$, for every $w \in W'$.

It is easy to check that the constructed model $\mathcal{M}$ satisfies every constraint on NCL-models and $\mathcal{M}, x \models tr_0(\varphi)$.

■

5 Translating Coalition Logic into NCL

First, we show a theorem in NCL which generalizes (AIA_1) from individuals to coalitions, and that will be instrumental later in the proof of superadditivity in Theorem 16.

LEMMA 15. $\vdash_{\mathsf{NCL}} \langle\emptyset\rangle[J_0]\varphi_0 \wedge \langle\emptyset\rangle[J_1]\varphi_1 \to \langle\emptyset\rangle([J_0]\varphi_0 \wedge [J_1]\varphi_1)$ *for* $J_0 \cap J_1 = \emptyset$.

PROOF. Suppose $J_0 \cap J_1 = \emptyset$. We establish the following deduction:

1. $\langle\emptyset\rangle[J_0]\varphi_0 \to \langle J_1\rangle\langle J_0\rangle[J_0]\varphi_0$ by Lemma 9
2. $\langle\emptyset\rangle[J_0]\varphi_0 \to \langle J_1\rangle[J_0]\varphi_0$ from previous line by S5($[J_0]$)
3. $\langle\emptyset\rangle[J_0]\varphi_0 \wedge [J_1]\varphi_1 \to \langle J_1\rangle[J_0]\varphi_0 \wedge [J_1][J_1]\varphi_1$ from previous line by S5($[J_1]$)
4. $\langle\emptyset\rangle[J_0]\varphi_0 \wedge [J_1]\varphi_1 \to \langle J_1\rangle([J_0]\varphi_0 \wedge [J_1]\varphi_1)$ from previous line by S5($[J_1]$)

5. $\langle\emptyset\rangle(\langle\emptyset\rangle[J_0]\varphi_0 \wedge [J_1]\varphi_1) \rightarrow \langle\emptyset\rangle\langle J_1\rangle([J_0]\varphi_0 \wedge [J_1]\varphi_1)$
from previous line by $[\emptyset]$-necessitation and $\mathrm{K}([\emptyset])$

6. $\langle\emptyset\rangle[J_0]\varphi_0 \wedge \langle\emptyset\rangle[J_1]\varphi_1 \rightarrow \langle\emptyset\rangle\langle J_1\rangle([J_0]\varphi_0 \wedge [J_1]\varphi_1)$
from previous line by $\mathrm{S5}([\emptyset])$

7. $\langle\emptyset\rangle[J_0]\varphi_0 \wedge \langle\emptyset\rangle[J_1]\varphi_1 \rightarrow \langle\emptyset\rangle([J_0]\varphi_0 \wedge [J_1]\varphi_1)$
from previous line by (Mon) and $\mathrm{S5}([\emptyset])$

∎

Now we give the following translation from Coalition Logic to NCL.

$$\begin{array}{lcl} tr(p) & = & p \\ tr(\langle\!\langle J \rangle\!\rangle\varphi) & = & \langle\emptyset\rangle[J]\mathbf{X}tr(\varphi) \end{array}$$

and homomorphic for the Boolean operators.

THEOREM 16. *If φ is a theorem of CL then $tr(\varphi)$ is a theorem of NCL.*

PROOF. First, the translations of the CL axiom schemas are valid:

- $tr(\neg\langle\!\langle J \rangle\!\rangle\bot) = \neg\langle\emptyset\rangle[J]\mathbf{X}\bot$

  By $\mathrm{D}(\mathbf{X})$, $\vdash_{\mathsf{NCL}} \mathbf{X}\bot \leftrightarrow \bot$. By $\mathrm{S5}([J])$, $\vdash_{\mathsf{NCL}} [J]\bot \leftrightarrow \bot$. It remains to prove that $\vdash_{\mathsf{NCL}} \neg\langle\emptyset\rangle\bot$, which follows from $\mathrm{S5}([\emptyset])$.

- $tr(\langle\!\langle J \rangle\!\rangle\top) = \langle\emptyset\rangle[J]\mathbf{X}\top$

  By $\mathrm{K}(\mathbf{X})$, $\vdash_{\mathsf{NCL}} \mathbf{X}\top \leftrightarrow \top$. By $\mathrm{S5}([J])$, $\vdash_{\mathsf{NCL}} [J]\top \leftrightarrow \top$. Finally, by $\mathrm{S5}([\emptyset])$, $\vdash_{\mathsf{NCL}} \langle\emptyset\rangle\top$.

- $tr(\neg\langle\!\langle\emptyset\rangle\!\rangle\neg\varphi \rightarrow \langle\!\langle AGT \rangle\!\rangle\varphi) = \neg\langle\emptyset\rangle[\emptyset]\mathbf{X}\neg tr(\varphi) \rightarrow \langle\emptyset\rangle[AGT]\mathbf{X}tr(\varphi)$.

  As $\vdash_{\mathsf{NCL}} [AGT]\psi \leftrightarrow \psi$ by $\mathrm{Triv}(AGT)$, and as $\vdash_{\mathsf{NCL}} \langle\emptyset\rangle[\emptyset]\psi \leftrightarrow [\emptyset]\psi$ by $\mathrm{S5}([\emptyset])$, the translation of (N) is equivalent to $\neg[\emptyset]\mathbf{X}\neg tr(\varphi) \rightarrow \langle\emptyset\rangle\mathbf{X}tr(\varphi)$. This is again equivalent to $\langle\emptyset\rangle\neg\mathbf{X}\neg tr(\varphi) \rightarrow \langle\emptyset\rangle\mathbf{X}tr(\varphi)$ which is proved a theorem from $\mathrm{Det}(\mathbf{X})$.

- $tr(\langle\!\langle J \rangle\!\rangle(\varphi\wedge\psi) \rightarrow \langle\!\langle\varphi\rangle\!\rangle\wedge\langle\!\langle J \rangle\!\rangle\psi) = \langle\emptyset\rangle[J]\mathbf{X}(tr(\varphi)\wedge tr(\psi)) \rightarrow \langle\emptyset\rangle[J]\mathbf{X}tr(\varphi)\wedge \langle\emptyset\rangle[J]\mathbf{X}tr(\psi)$

  First, $\vdash_{\mathsf{NCL}} \mathbf{X}(tr(\varphi) \wedge tr(\psi)) \rightarrow \mathbf{X}tr(\varphi) \wedge \mathbf{X}tr(\psi)$ by $\mathrm{K}(\mathbf{X})$. We have $\vdash_{\mathsf{NCL}} [J]\mathbf{X}(tr(\varphi)\wedge tr(\psi)) \rightarrow [J]\mathbf{X}tr(\varphi)\wedge[J]\mathbf{X}tr(\psi)$ by $[J]$-necessitation and we conclude by $[\emptyset]$-necessitation.

- $tr(\langle\!\lbrack J_1\rbrack\!\rangle\varphi\wedge\langle\!\lbrack J_2\rbrack\!\rangle\psi \rightarrow \langle\!\lbrack J_1\cup J_2\rbrack\!\rangle(\varphi\wedge\psi)) = \langle\emptyset\rangle[J_1]\mathbf{X}tr(\varphi)\wedge\langle\emptyset\rangle[J_2]\mathbf{X}tr(\psi) \rightarrow \langle\emptyset\rangle[J_1 \cup J_2]\mathbf{X}(tr(\varphi) \wedge tr(\psi))$

  Assume $J_1\cap J_2 = \emptyset$. The proof that $\vdash_{\mathsf{NCL}} \langle\emptyset\rangle[J_1]\mathbf{X}tr(\varphi)\wedge\langle\emptyset\rangle[J_2]\mathbf{X}tr(\psi) \rightarrow \langle\emptyset\rangle[J_1 \cup J_2]\mathbf{X}(tr(\varphi) \wedge tr(\psi))$ is done by the following steps:

  1. $\langle\emptyset\rangle[J_1]\mathbf{X}tr(\varphi)\wedge\langle\emptyset\rangle[J_2]\mathbf{X}tr(\psi) \rightarrow \langle\emptyset\rangle([J_1]\mathbf{X}tr(\varphi)\wedge[J_2]\mathbf{X}tr(\psi))$ by Lemma 15
  2. $[J_1]\mathbf{X}tr(\varphi) \wedge [J_2]\mathbf{X}tr(\psi) \rightarrow [J_1 \cup J_2]\mathbf{X}tr(\varphi) \wedge [J_1 \cup J_2]\mathbf{X}tr(\psi)$ by (Mon)
  3. $\langle\emptyset\rangle([J_1]\mathbf{X}tr(\varphi) \wedge [J_2]\mathbf{X}tr(\psi)) \rightarrow \langle\emptyset\rangle([J_1 \cup J_2](\mathbf{X}tr(\varphi) \wedge \mathbf{X}tr(\psi))$ from previous line and $[\emptyset]$-necessitation
  4. $\langle\emptyset\rangle[J_1]\mathbf{X}tr(\varphi) \wedge \langle\emptyset\rangle[J_2]\mathbf{X}tr(\psi) \rightarrow \langle\emptyset\rangle[J_1 \cup J_2]\mathbf{X}(tr(\varphi) \wedge tr(\psi))$ from line 1 and 3 by standard modal principles

Clearly the translation of modus ponens preserves validity. To prove that the translation of CL's (RE) preserves validity suppose $tr(\varphi \leftrightarrow \psi) = tr(\varphi) \leftrightarrow tr(\psi)$ is a theorem of NCL. We have to prove that $tr(\langle\!\lbrack J\rbrack\!\rangle\varphi \leftrightarrow \langle\!\lbrack J\rbrack\!\rangle\psi) = \langle\emptyset\rangle[J]\mathbf{X}tr(\varphi) \leftrightarrow \langle\emptyset\rangle[J]\mathbf{X}tr(\psi)$ is a theorem of NCL. This follows from the theoremhood of $tr(\varphi) \leftrightarrow tr(\psi)$ by standard modal principles. ■

LEMMA 17. *Let $M = ((S, E), V)$ be a coalition model. let selec be a function associating to each state s in S a choice profile $selec(s)$ in $\mu_E(AGT, s)$.*[3] *Let the tuple (W, R, F_X, π) be constructed as follows:*

- $W = \{\langle s, Y\rangle \mid s \in S, Y \in \mu_E(AGT, s)\}$
- $R_J = \{(\langle s, Y\rangle, \langle s, Y'\rangle) \mid$ *there is* $Z \in \mu_E(J, s)$ *such that* $Y \cup Y' \subseteq Z\}$
- $F_X(\langle s, Y\rangle) = \langle s', Z\rangle$, *where* $Y = \{s'\}$ *and* $Z = selec(s')$
- $\pi(\langle s, X\rangle) = V(s)$

Then (W, R, F_X, π) is a model of NCL.

PROOF. The proof consists in checking that the constructed model satisfies every constraint on NCL models. Note that we are permitted to define F_X this way because of Proposition 8. ■

THEOREM 18. *If φ is CL-satisfiable then $tr(\varphi)$ is satisfiable in NCL.*

PROOF. Given a coalition model $M = ((S, E), V)$ we construct a model $\mathcal{M}_{NCL} = (W, R, F_X, \pi)$ of NCL as in Lemma 17. We prove by structural

[3]Such a function exists by the axiom of choice.

induction that $M, s \models \varphi$ iff $\exists Y \in \mu_E(AGT, s)$ such that $\mathcal{M}_{NCL}, \langle s, Y\rangle \models tr(\varphi)$.

The cases of atoms and classical connectives are straightforward. We just consider the case of $\varphi = \langle\!\langle J \rangle\!\rangle \psi$.

1. Suppose, $M, s \models \langle\!\langle J \rangle\!\rangle \psi$. Then, there is $Z' \in E_s(J)$ such that for all $t \in Z', M, t \models \psi$. Thus, there is $Z \in \mu_E(J, s)$ such that for all $t \in Z$, $M, t \models \psi$. By induction hypothesis, for all $t \in Z$, $\mathcal{M}_{NCL}, \langle t, selec(t)\rangle \models tr(\psi)$.
2. By construction, for all $t \in Z, \forall Y \in \mu_E(AGT, s)$ such that $Y \subseteq Z, F_X(\langle s, Y\rangle) = \langle t, selec(t)\rangle$.
3. By (1) and (2) it follows that $\forall Y \in \mu_E(AGT, s)$ such that $Y \subseteq Z$, $\mathcal{M}_{NCL}, \langle s, Y\rangle \models \mathbf{X} tr(\psi)$, and thus, since $Z \in \mu_E(J, s)$, it follows that $\exists Y \subseteq Z$ such that $\mathcal{M}_{NCL}, \langle s, Y\rangle \models [J]\mathbf{X} tr(\psi)$.
4. And then, there is $Y' \in \mu_E(AGT, s)$ such that $\mathcal{M}_{NCL}, \langle s, Y'\rangle \models \langle\emptyset\rangle[J]\mathbf{X} tr(\psi)$.

The other direction of the induction step is verified by reverse arguments. ■

COROLLARY 19. *φ is a theorem of CL iff $tr(\varphi)$ is a theorem of NCL.*

PROOF. The left-to-right direction is Theorem 16. The right-to-left direction follows from Pauly's completeness result for Coalition Logic and Theorem 18. ■

6 Decidability and complexity of NCL

In this section, we study the satisfiability problem of an NCL-formula.[4] We first study the fragment of NCL without time.

6.1 NCL without time

In the remaining, we call NCLwt(n) the particular instance of NCL with n agents and without the temporal operator $\mathbf{X}$. NCLwt(n) is the fragment of NCL(n) defined by the BNF:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid [J]\varphi$$

where J ranges over the set of subsets of $\{0, ..., n-1\}$.

DEFINITION 20. A *NCLwt-model* is a tuple $\mathcal{M} = (W, R, \pi)$ where:

[4] It was the subject of [Sch07] (in French).

- W is a set of contexts;
- R is a collection of equivalence relations R_J (one for every coalition $J \subseteq AGT$) such that:
  - $R_{J_1 \cup J_2} \subseteq R_{J_1} \cap R_{J_2}$
  - $R_{\emptyset} \subseteq R_J \circ R_{AGT \setminus J}$
  - $R_{AGT} = Id$
- $\pi : W \to 2^{Prop}$ is a valuation function.

PROPOSITION 21. *NCLwt(n) is both sound and complete w.r.t. to the class of NCLwt-models where $R_{\emptyset}$ is the universal relation.*

PROOF. Soundness is obtained by a routine argument while completeness is immediate from Sahlqvist's theorem. ■

We now define filtration.

DEFINITION 22. Let $SF(\Phi)$ be the set of all subformulas of an NCLwt-formula Φ.

The set of formulas we use to filter is defined as follows:

DEFINITION 23. Given an NCLwt-formula Φ, let $Cl(\Phi)$ be the set $SF(\Phi) \cup \{[J]\varphi \mid J \subseteq AGT, \varphi \in SF(\Phi)\}$.

PROPOSITION 24. $card(Cl(\Phi)) \leq 2^{card(AGT)} \times |\Phi|$, *where $|\Phi|$ is the length of the formula Φ.*

The filtered model is defined as follows:

DEFINITION 25. Let $M = (W, \{R_J, J \subseteq AGT\}, \pi)$ be an NCLwt-model. We define the equivalence relation $\leftrightsquigarrow^{Cl(\Phi)}$ on W as follows:

$$\forall x, y \in W, x \leftrightsquigarrow^{Cl(\Phi)} y \text{ iff } (\forall \varphi \in Cl(\Phi), M, x \models \varphi \Leftrightarrow M, y \models \varphi)$$

DEFINITION 26. Given $M = (W, \{R_J, J \subseteq AGT\}, \pi)$ an NCLwt-model and Φ an NCLwt-formula, we define the *filtered model* $M' = (W', \{R'_J, J \subseteq AGT\}, \pi')$ as follows:

- The worlds in W' are the equivalence classes of worlds in W under the relation $\leftrightsquigarrow^{Cl(\Phi)}$;
- For all $J \subseteq AGT$, we define R'_J as $|x| R'_J |y|$ iff $\forall \varphi \in SF(\Phi), \forall J' \subseteq J, M, x \models [J']\varphi$ iff $M, y \models [J']\varphi$;

- For all $|x|_{\leftrightsquigarrow} \in W'$ and for all atomic formulas p in $SF(\Phi)$, $p \in \pi'(|x|_{\leftrightsquigarrow})$ iff $p \in \pi(x)$.

We now study the properties of the filtered model.

PROPOSITION 27. *M' contains at most $2^{2^{card(AGT)} \times |\Phi|}$ worlds.*

PROOF. We use Proposition 24 and the fact that

$$\begin{array}{lll} W' & \rightarrow & 2^{Cl(\Phi)} \\ |x|_{\leftrightsquigarrow} & \mapsto & \{\varphi \in Cl(\Phi) \mid M, x \models \varphi\} \end{array}$$

is a well-defined injective application. ■

PROPOSITION 28. *If M is an NCLwt-model such that $R_\emptyset = W \times W$ and Φ any formula, then the model M' filtered by Φ is an NCLwt-model.*

PROPOSITION 29. *Given a formula Φ, for all φ in $SF(\Phi)$ and for all $x \in W$, we have $M', |x|_{\leftrightsquigarrow} \models \varphi \Leftrightarrow M, x \models \varphi$.*

PROOF. By induction on φ. ■

THEOREM 30. *An $\mathsf{NCLwt}(n)$-formula Φ is satisfiable iff it is satisfiable in a model with $2^{2^{card(AGT)} \times |\Phi|}$ worlds.*

PROOF. By Propositions 21, 27, 28 and 29. ■

THEOREM 31. *The problem of deciding satisfiability of an $\mathsf{NCLwt}(1)$-formula is NP-complete.*

PROOF. $\mathsf{NCLwt}(1)$ is nothing but S5 because:

- $\langle\emptyset\rangle$ is S5-operator;
- $\langle AGT\rangle$ is a trivial operator since $\langle AGT\rangle\varphi \leftrightarrow \varphi$, and can thus be eliminated.

■

PROPOSITION 32. *If $n \geq 2$ then the problem of deciding satisfiability of an $\mathsf{NCLwt}(n)$-formula is is in NEXPTIME.*

PROOF. According to Theorem 30, we can test the satisfiability of a formula by examining all models with $2^{2^{card(AGT)} \times |\varphi|}$ worlds. ■

We are going to compare $\mathsf{NCLwt}(n)$ and $[S5; S5]$ defined as follows:

DEFINITION 33. $[S5; S5]$ is the modal logic defined by:

- a language with two modal operators $\langle 1\rangle$ and $\langle 2\rangle$;
- an axiomatics with $S5(\langle 1\rangle)$, $S5(\langle 2\rangle)$ and the axiom of permutation $[1][2]\varphi \leftrightarrow [2][1]\varphi$.

Note that the Church-Rosser axiom $\langle 1\rangle[2]\varphi \to [2]\langle 1\rangle\varphi$ is a $[S5; S5]$-theorem.

PROPOSITION 34. *$[S5; S5]$ is characterized by the class of frames $F = (W, R_1, R_2)$ such that:*

- *R_1 and R_2 are equivalence relations ;*
- *$R_1 \circ R_2 = R_2 \circ R_1 = W \times W$.*

THEOREM 35 ([GKWZ03]). *The problem of deciding satisfiability of an $[S5; S5]$-formula is NEXPTIME-hard.*

PROPOSITION 36. *If $n \geq 2$ then $\mathsf{NCLwt}(n)$ is a conservative extension of $[S5; S5]$.*

PROOF. We define $tr : \mathcal{L}_{S5^2} \to \mathcal{L}_{NCLwt}$ which replaces the two operators $\langle 1\rangle$ and $\langle 2\rangle$ of $[S5; S5]$ by $\langle\{1\}\rangle$ and $\langle\{2\}\rangle$ respectively.

First, the reader may easily verify that $\vdash_{[S5;S5]} \varphi$ implies $\vdash_{NCLwt} tr(\varphi)$. Second, if φ is a satisfiable $[S5; S5]$ formula, there is a $[S5; S5]$-model (M, x), where we suppose that $R_1 \circ R_2 = W \times W$, such that $M, x \models \varphi$. We extend M to an NCLwt-model M' by stipulating:

- If $1 \in J$ and $2 \in J$ then $R_J = Id_W$;
- If $1 \in J$ and $2 \notin J$ then $R_J = R_1$;
- If $1 \notin J$ and $2 \in J$ then $R_J = R_2$;
- If $1 \notin J$ and $2 \notin J$ then $R_J = W \times W$.

It is straightforward to check that M' is an NCLwt-model and that $M', x \models tr(\varphi)$. ■

COROLLARY 37. *If $n \geq 2$ then the problem of deciding satisfiability of an $\mathsf{NCLwt}(n)$-formula is NEXPTIME-hard.*

PROOF. From Theorem 35 and Proposition 36. ■

THEOREM 38. *If $n \geq 2$ then the problem of deciding satisfiability of an $\mathsf{NCLwt}(n)$-formula is NEXPTIME-complete.*

PROOF. From Proposition 32 and Corollary 37. ■

```
function recursive NCL(n)-SAT-ND(φ)
    if φ does not contain any X then
        return NCLwt-SAT-ND(φ)
    else
        φ' := Freeze(φ)
        J := Xat(φ')
        if NCLwt-SAT-ND(φ') = UNSATISFIABLE then
            return UNSATISFIABLE
        (M = (W, {R_J}, π), r) := NCLwt-GiveModel-ND(φ')
        for y ∈ W,
            ψ_y := ⋀_{Xψ∈J/p_Xψ∈π(y)} ψ ∧ ⋀_{Xψ∈J/p_Xψ∉π(y)} ¬ψ
            if NCL(n)-SAT-ND(ψ_y) = UNSATISFIABLE then
                return UNSATISFIABLE
        endFor
        return SATISFIABLE
    endIf
endFunction
```

Figure 1. NCL(n)-SAT-ND

6.2 NCL(n)

For NCL(n), i.e. with the temporal operator $\mathbf{X}$, it is difficult to apply filtration directly because we cannot assume that $R_\emptyset$ is the universal relation any more. First let us introduce the notion of a frozen formula.

DEFINITION 39. Let $\mathbf{X}at(\varphi)$ be the set of all the subformulas of φ of the form $\mathbf{X}\psi$ that are not proper subformulas of some other subformula $\mathbf{X}\varphi_1$.

DEFINITION 40. Let $Freeze(\varphi)$ be the formula φ where all subformulas $\mathbf{X}\psi \in \mathbf{X}at(\varphi)$ are replaced by a new atomic formula $p_{\mathbf{X}\psi}$.

EXAMPLE 41. For $\varphi = \mathbf{X}p \vee \langle\{1\}\rangle(p \wedge \mathbf{X}(\langle\{2\}\rangle p \vee \mathbf{X}q) \wedge \langle\{2,4\}\rangle\mathbf{X}\mathbf{X}q)$, we have:

- $\mathbf{X}at(\varphi) = \{\mathbf{X}p, \mathbf{X}(\langle\{2\}\rangle p \vee \mathbf{X}q), \mathbf{X}\mathbf{X}q\}$;
- $Freeze(\varphi) = p_{\mathbf{X}p} \vee \langle\{1\}\rangle(p \wedge p_{\mathbf{X}(\langle\{2\}\rangle p \vee \mathbf{X}q)} \wedge \langle\{2,4\}\rangle p_{\mathbf{X}\mathbf{X}q})$.

Figure 1 shows a non-deterministic algorithm to decide the satisfiability problem of an NCL(n)-formula φ. The procedure NCL(n)-SAT-ND uses two sub-routines:

- NCLwt-SAT-ND is a non-determistic decision procedure for NCLwt-SAT;

- NCLwt-GiveModel-ND(φ') nondeterministically chooses an NCLwt(n)-pointed-model (M, r) such that $M, r \models \varphi'$.

As usual, an NCLwt(n)-pointed-model is a pair (M, r) where M is an NCLwt(n)-model and r is a context of M.

Both NCLwt-SAT-ND and NCLwt-GiveModel-ND(φ') are optimal, that is to say:

- in the mono-agent case, they run in a polynomial space. In particular, NCLwt-GiveModel-ND(φ') returns a model of polynomial size;
- if there are more than two agents ($n \geq 2$), they run in exponential time. NCLwt-GiveModel-ND(φ') returns a model with $2^{2^n \times |\varphi'|}$ worlds.

The procedure goes along the following idea: if an NCL-formula φ does not contain any $\mathbf{X}$ symbol, then we can immediately use the first NCLwt-SAT-ND procedure. Else we begin by treating the satisfiability of the NCLwt-formula $Freeze(\varphi)$. If it is satisfiable, we choose an NCLwt-model M, r such that $M, r \models Freeze(\varphi)$. Then we try to know if the valuation of the propositions in $\mathbf{X}at(\varphi)$ is consistent in every world y of the model M. This is why we test whether ψ_y is NCL-satisfiable for every world y of the model M by recursive calls to NCL(n)-SAT-ND.

THEOREM 42. *NCL-SAT-ND terminates for all φ.*

PROOF. By induction on the modal degree w.r.t. the $\mathbf{X}$ operator. ∎

THEOREM 43. *NCL-SAT-ND(φ) returns SATISFIABLE iff φ is satisfiable.*

PROOF.

The proof is done by induction.

$\boxed{\Rightarrow}$ We can construct an NCL-model for φ by gluing together the model M that is built in NCLwt-GiveModel-ND with an NCL-model M_y for each ψ_y (as exemplified in Figure 2).

$\boxed{\Leftarrow}$ If φ is satisfiable then there is an NCL-pointed-model (N, x) which satisfies φ. Then we proceed as follows (see Figure 3).

1. We extract from (N, x) an NCLwt(n) model G for $Freeze(\varphi)$.
2. We filter G: we obtain G'.
3. One execution of NCLwt-SAT-ND is such that $M = G'$. We then take into account that any ψ_y is satisfiable (in (N, t) where t is a world of N).

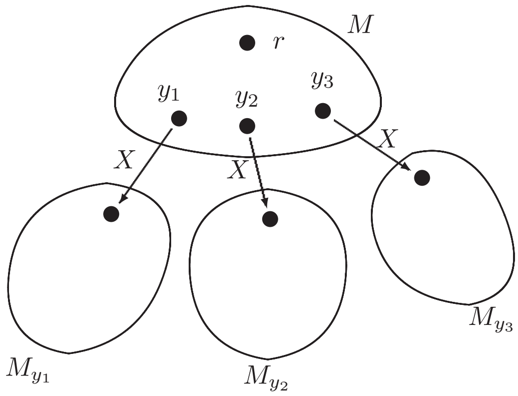

Figure 2. Construction of the NCL-model ($\boxed{\Rightarrow}$-sense proof of Theorem 43).

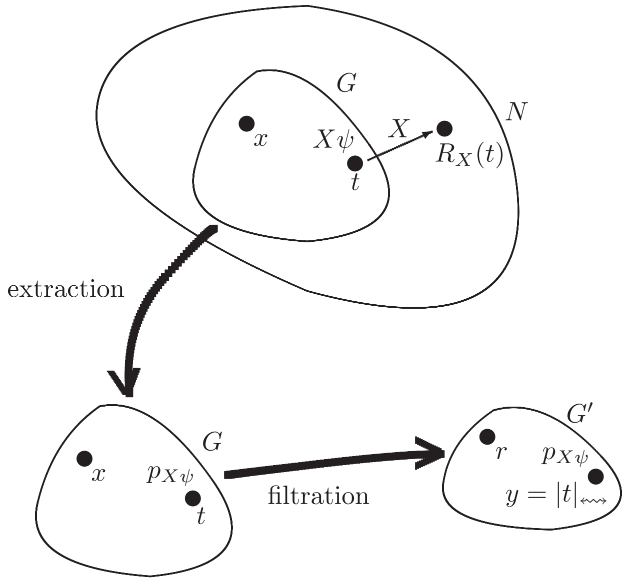

Figure 3. Picture explaining the $\boxed{\Leftarrow}$-sense proof of Theorem 43.

Finally, NCL-SAT-ND(φ) returns SATISFIABLE.

■

We now state the complexity of NCL(1).

PROPOSITION 44. *In the mono-agent setting, the problem of deciding the satisfiability of an NCL-formula is in PSPACE.*

PROOF. By induction we can prove that an execution of NCL-SAT-ND uses polynomial space. We take into account that:

- NCLwt-SAT-ND and NCLwt-GiveModel-ND run in polynomial space;
- M contains a polynomial number of worlds.

Then we use that NPSPACE = PSPACE (Savitch's theorem). ■

PROPOSITION 45. *In the mono-agent setting, we have a polynomial reduction from the satisfiability problem of a K-formula to the satisfiability problem of an NCL-formula by the following translation $tr : \mathcal{L}_K \to \mathcal{L}_{NCL}$ defined by: $tr(\Box\varphi) = \mathbf{X}[\emptyset]tr(\varphi)$.*

COROLLARY 46. *In the mono-agent setting, the problem of deciding the satisfiability of an NCL-formula is PSPACE-hard.*

THEOREM 47. *In the mono-agent setting, the problem of deciding the satisfiability of an NCL-formula is PSPACE-complete.*

PROOF. From Proposition 44 and Corollary 46. ■

We finally give the complexity of the satisfiability problem of an NCL(n)-formula for $n \geq 2$.

PROPOSITION 48. *The problem of deciding the satisfiability of an NCL(n)-formula is in NEXPTIME.*

PROOF. By induction, we can prove that an execution of NCL-SAT-ND uses exponential time. We take into account that:

- NCLwt-SAT-ND and NCLwt-GiveModel-ND run in exponential time;
- M contains an exponential number of worlds.

■

PROPOSITION 49. *NCL is a conservative extension of NCLwt.*

PROOF. First, an NCLwt proof is an NCL proof. Second, if φ is a satisfiable NCLwt-formula in an NCLwt-model (M, x), we extend M to M' by adding $F_X = Id_W$. We have $M', x \models \varphi$. ■

COROLLARY 50. *In presence of at least two agents, the problem of deciding the satisfiability of an* NCL*-formula is NEXPTIME-hard.*

PROOF. From Corollary 37 and Proposition 49. ■

THEOREM 51. *In presence of at least two agents, the problem of deciding the satisfiability of an* NCL*-formula is NEXPTIME-complete.*

PROOF. From Proposition 48 and Corollary 50. ■

7 Discussion

Coalition Logic is basically a logic of ability, in the sense that its main operator formalizes sentences of the form "agent i is able to ensure φ". As we have seen, NCL embeds CL and is of course suitable for such kind of reasoning about abilities of agents and coalitions. However, the introduction of a STIT-style operator is a move to more expressivity.

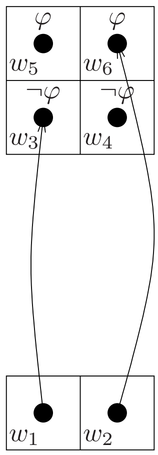

Figure 4. Representation of an NCL-model with two moments and two agents: i chooses the columns ($R_{\{i\}}$) and j chooses the rows ($R_{\{j\}}$). The *grand* coalition can determine a unique outcome: $R_{\{i,j\}} = Id$, is represented by the 'small squares'. Nature ($\emptyset$) cannot distinguish outcomes of a same moment: $R_\emptyset = R_{\{i\}} \circ R_{\{j\}}$ is represented by the 'big boxes'. Arrows are F_X transitions.

Authors in logics of action have often been interested in the notion of 'making do'. It can be linked to the idea of an agent having the *power over* another agent [Cas03]. On the model of Figure 4, it is easy to check that the formula $\langle\emptyset\rangle[\{i\}]\mathbf{X}[\{j\}]\varphi$ is satisfied at w_1 and w_2. A direct reading of this formula is "agent i sees to it that next, agent j sees to it that φ".

For example, in an organizational or normative setting, it in fact reflects adequately the agentive component of a *delegation*. As an illustration of our logic, we see here how NCL can grasp tighter notions of ability than Coalition Logic.

Chellas's STIT logic has some annoying properties: if we try to model influence of an agent on an other, we are inclined to state it via the formula $[i][j]\varphi$. It is nevertheless equivalent to $\Box\varphi$. (Recall the historical necessity operator in Section 4.) Hence, in this logic, an agent can force another agent to do something if and only if this something is settled. We must admit this is a poor notion of influence.

In previous attempts to extend straightforwardly the logic of Chellas's stit with a 'next' operator ([BHT06b]), the formula $[\{i\}]\mathbf{X}[\{j\}]\varphi \rightarrow \mathbf{X}\Box\varphi$ was valid. It means that if i forces that next j ensures φ then next, φ is inevitable. Inserting an $\mathbf{X}$ operator between the agent's actions gives us a refined notion of influence. Still, it is not completely satisfying, since it suggests that an agent influences another agent j to do φ by forcing the world to be at a moment where φ is settled. Since an agent at a moment sees to everything being historically necessary (in formula: for every i, $\Box\varphi \rightarrow [i]\varphi$), it means that an agent i influences an agent to do φ if and only if it influences every agent to do φ, i included.

On the contrary, the following formula is not a theorem of NCL:

$$[\{i\}]\mathbf{X}[\{j\}]\varphi \rightarrow \mathbf{X}[\emptyset]\varphi.$$

In particular in the model of Figure 4, the following formulas are true at w_1 and w_2:

- $\langle\emptyset\rangle[\{i\}]\mathbf{X}[\{j\}]\varphi$
- $\langle\emptyset\rangle[\{i\}]\mathbf{X}[\{j\}]\neg\varphi$

It somewhat grasps the fact that agent i controls the truth value of φ by exerting influence on j. An interesting account of similar concepts but focused on propositional control is given by [vdHW05]. Of course, CL 'fused' operator is not designed for those issues, and Coalition Logic is not suitable for modeling the notion of *power over*.

Even though our quick study does not permit to prove that NCL is indeed a good logic to reason about influence, we think that the consistency of $\langle\emptyset\rangle[\{i\}]\mathbf{X}([\{j\}]\varphi\wedge\langle\emptyset\rangle[\{j\}]\neg\varphi)$ which is at first sight a drawback, is in fact an interesting property: an agent can force an agent j to ensure φ even if j would also be able to ensure $\neg\varphi$. It somewhat leaves some place to indeterminism and unsuccessful delegations. What should constrain a delegated agent is not physics but norms. If one wants to rule that property out, one could simply release Det($\mathbf{X}$) and add the axiom schema $\mathbf{X}\varphi \rightarrow \mathbf{X}[\emptyset]\varphi$. The nature of time in NCL is simply a very convenient one for embedding CL and is amenable at will. We believe it particularly deserves a work effort in the future.

8 Conclusion

To conclude, we have investigated the properties of a normal modal logic version of Pauly's Coalition Logic that we call NCL. We have shown that due to its richer language it is strictly more complex than the latter: satisfiability checking is NEXPTIME-complete (for more than two agents).

We think that the versatility of NCL models allows for smoothness in modeling. The information 'contained' in a context, viz. the physical description of the world *and* the actual strategy profile of agents may permit to capture fine-grained notions relevant for multi-agent systems via Kripke models in the realm of normal modal logics.

We believe that the present framework opens new perspectives on a proof theoretic investigation of logics of agency.

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