A Logic of Games and Propositional Control

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ABSTRACT

We present a logic for reasoning about strategic games. The logic is a modal formalism, based on the Coalition Logic of Propositional Control, to which we add the notions of outcomes and preferences over outcomes. We study the underlying structure of powers of coalitions as they are expressed in their effectivity function, and formalise a collection of solution concepts. We provide a sound and complete axiomatisation for the logic, and we demonstrate its features by applying it to some problems from social choice theory.

Categories and Subject Descriptors

I.2.4 [**Artificial Intelligence**]: Knowledge Representation Formalisms and Methods—*modal logic*; I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*multiagent systems*

General Terms

Theory

Keywords

propositional control, modal logic, strategic games, effectivity functions, solution concepts, game solvability

1. INTRODUCTION

Game theory (GT) [17] and social choice theory (SCT) [7] have come to be seen as topics of major importance for computer science, since they focus on the study of interaction and protocols from an incentive-based perspective. *Social software* [18] aims to give social procedures a theory analogous to the formal theories for computer algorithms, e.g., program correctness or analysis of programs. One aspect of *game logics* [22] is to study those theories with logical tools.

The aim of this paper is to present and investigate a formal logical framework for representing multi-agent systems combining action and preferences of agents, in a way consistent with game and social choice theory. In particular, we will focus on models of interaction called *strategic games*. In order to fix terminology, we review some relevant concepts from game theory.

Definition 1 (strategic game form). *A* strategic game form *is a tuple* $\langle N, (A_i), K, o\rangle$ *where:*

- N *is a finite nonempty set of* players *(or agents);*
- A_i *is a finite nonempty set of* actions *for each player* $i \in N$*;*
- K *is a finite nonempty set of outcomes;*
- $o : \times_{i\in N} A_i \longrightarrow K$ *determines an outcome for every combination of actions.*

A strategic game form is sometimes called a *mechanism*. It specifies the agents taking part of the game, their available actions and the protocol. Next, we need preferences which will give the players the incentives for taking an action.

Definition 2 (preference relation). *A* preference relation $\succeq$ *over* K *is a total, transitive and reflexive binary relation over* K*.*

We can now see a strategic game as basically the composition of a strategic game form with a collection of preference relations (one for every agent) over the set of outcomes.

Definition 3 (strategic game). *A* strategic game *is a tuple* $\langle N, (A_i), K, o, (\succeq_i)\rangle$ *where* $\langle N, (A_i), K, o\rangle$ *is a strategic game form, and for each player* $i \in N$, $\succeq_i$ *is a* preference relation *over* K*.*

We refer to a collection $(a_i)_{i\in N}$, consisting of one action for every agent in N, as an *action profile*. Given an action profile a, we denote by a_i the action of the player i, and by a_{-i} the action profile of the coalition $N \setminus \{i\}$. We write a_C for the *coalitional actions* that are members of $A_C = \times_{j\in C} A_j$ for any $C \subseteq N$. We refer to a collection $(\succeq_i)_{i\in N}$ of preferences as a *preference profile*.

A solution concept defines for every game a set of action profiles, intuitively corresponding to action profiles that may be played through rational action. Exactly which solution concept is used depends on the application at hand.

Definition 4 (solution concept). *A* solution concept *is a function that maps a strategic game form* $\langle N, (A_i), K, o\rangle$ *and a preference profile over* K *to a subset of the action profiles in* A_N*.*

Several modal logics have been successfully applied for representing some aspects of game theory and social choice theory. They usually propose an isolated study of one aspect of rational action such as actions, knowledge, preferences, etc.

Modal logics of preference date back to the work of Halldén and von Wright (see [14]). More recently, the paradigm of game logics has triggered renewed interest. For instance, the authors of [23] present several logics of preference whose expressivity is of interest in social choice theory, e.g., the logic of preferences *ceteris paribus* or several interpretations of a binary preference operator. The work in [3] contains a proposal of a logic for preference aggregation and is able to formalise in a logical language some important results of SCT such as Arrow's theorem.

Cite as: A Logic of Games and Propositional Control, Nicolas Troquard, Wiebe van der Hoek and Michael Wooldridge, *Proc. of 8th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2009)*, Decker, Sichman, Sierra and Castelfranchi (eds.), May, 10–15, 2009, Budapest, Hungary, pp. XXX-XXX.
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A formalisation of actions, on the other hand, makes it possible to specify several issues of interest in a strategic context: e.g., the means by which a player interacts in the system, the outcomes of a combination of actions or the powers of the players. Concerning game logics of action, *Coalition Logic* (CL) [19] is conceptually important, as it characterises the class of powers of coalitions underlying strategic game forms. Moreover, [13] presents an extension of CL to a particular class of game forms that admit a Nash equilibrium (the celebrated solution concept) whatever preferences of players are.

Although CL and related logics (like ATL [5]) study the power of agents, they do not explain where this power arises from. *Coalition Logic of Propositional Control* (CL-PC) [25] makes the origin of these powers concrete: the idea is that every agent (and, by inheritance, every coalition) controls a number of propositional variables, in the sense that they can freely choose to make these variables true or false. So, in CL-PC, doing an action corresponds to choosing a valuation for the propositions under your control, and this is a very natural setting when one is interested in building software agents, and likes to think of those agents as setting and unsetting bits in some digital system. In fact, in implemented verification systems for agents, individual powers are specified by allocating agents propositions. (For instance, in the model checker MOCHA for ATL ([4]), the keyword `controls` indicates precisely for each participant which propositional variables it is able to determine the value of). It is not unnatural to also conceive of actions or moves in a game (like "player i is going to a Bach concert") as the toggling of atomic variables b_i.

CL-PC is then a logic that deals with agency and contingent ability of players in strategic games. Every player controls the truth values of a particular set of atoms At_i. The set of possible valuations over At_i is easily understood as representing the set of actions in the repertoire of the player i.

Definition 5 (frames). *A* frame of propositional control *is a tuple* $\langle N, At, (At_i)\rangle$*, such that:*

- $N = \{1, 2, \dots n\}$ *is a nonempty finite set of players;*
- At *is a nonempty finite set of atoms;*
- At_i*, a subset of* At*, is the set of atoms controlled by agent i.*

We require that $At = At_1 \cup \dots \cup At_n$ *and* $At_i \cap At_j = \emptyset$ *for* $i \neq j$*.*

Every variable is controlled by one and only one agent: the sets At_i form a partition of At. We refer to At_C as the union of the controlled propositions At_i of every agent i in C.

Definition 6 (valuations). *Given a coalition C, a* C-valuation θ_C *is a function* $\theta_C : At_C \longrightarrow \{\mathtt{tt}, \mathtt{ff}\}$*. The function* π *allows us to reify these valuations in the object language.*

$$\pi(\theta_C) \triangleq \bigwedge_{p \in At_C, \theta_C(p)=\mathtt{tt}} p \wedge \bigwedge_{q \in At_C, \theta_C(q)=\mathtt{ff}} \neg q.$$

Game theoretically, a C-valuation (viz. a valuation of all the variables in At_C) can be identified with a coalitional action. Like an action profile, an N-valuation specifies the choice of every player.

We denote Θ the set of N-valuations. When it is clear from the context, we shall write θ instead of θ_N. Hence, θ_C can indeed be conceived of the restriction of θ to At_C. In our models, valuations θ_N will play the same role as states in Kripke models. Given θ and θ' in Θ, we write $\theta \equiv_C \theta'$ to mean $\theta_C = \theta'_C$. We shall sometimes slightly abuse notation and decompose a valuation θ. Let $\{C_1, \dots C_k\}$ a partition of N; we denote by θ_N the tuple $(\theta_{C_1}, \dots, \theta_{C_k})$.

CL-PC allows one to reason about strategic game *forms* when the set of outcomes is the set of N-valuations. Thus, there are two features of the games defined in Definition 3 that are missing: (i) we need to model the set K of outcomes; (ii) we need to model the preferences of players.

In this paper we propose to combine CL-PC with a modality of preferences. Combining modal logics is, of course, a non trivial task. For instance, CL-PC is in fact close to a *product of modal S5 logics*, and in a technical sense it is quite surprising that the logic is decidable (see a discussion in [10]). In our logic, as in GT, preferences are not directly over the N-valuations of CL-PC models, i.e., over action profiles, but rather over outcomes, a notion that we will add to those models. This has the advantage that we can model that agents have preferences over certain 'by-products' (outcomes) of their combined actions, and not directly about how these outcomes are being brought about.

The remainder is as follows. We present the *games of propositional control with consequences* (GPCC) in Section 2. We give an axiomatisation and prove that it determines the class of GPCC. In Section 3 we study some game and social choice theoretical aspects of the logic. We first study the structure of powers of coalitions, then we give a general formalisation of several solution concepts. We discuss two applications in Section 4. We briefly present model checking of game equilibria and we demonstrate how the logic can be employed in problems of game form solvability. We conclude in Section 5.

2. A LOGICAL CONCEPTUALISATION OF STRATEGIC GAMES

2.1 Language and semantics

The language $\mathcal{L}(N, At, K)$ is inductively defined by the following grammar:

$$\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond_C \varphi \mid \langle \preceq_i \rangle \varphi$$

where a is an atom of $At \cup K$, C is a coalition and i is a member of N.

$\Diamond_C \varphi$ reads that providing that the players outside C hold on with their current choice, the coalition C can ensure φ. $\langle \preceq_i \rangle \varphi$ reads that the player i prefers φ (or is indifferent) to the current state of affairs.

We have seen in the previous section how CL-PC can be seen as capturing a version of strategic forms. Now, an immediate way to model the set K of outcomes is to associate every element of K to a new atom. These features have to be fused with the frames of propositional control in order to obtain the models of our logic.

Definition 7. *A* game of propositional control with consequences *(notation: GPCC) is a tuple* $\langle N, At, (At_i), K, o, (\preceq_i)\rangle$*, such that:*

- $\langle N, At, (At_i)\rangle$ *is a frame of propositional control;*
- K *is a nonempty finite set of atoms such that* $At \cap K = \emptyset$*;*
- o *maps a* θ_N *valuation to an element of* K*;*
- $\preceq_i$ *is a preference relation over* K *for every agent i.*

We illustrate the integration of the set of outcomes by an example.

Example 1. *Consider a model* $\langle N, At, At_1, At_2, K, o, (\preceq_i)\rangle$ *with* $N = \{1, 2\}$*,* $At_1 = \{p_1, p_2\}$*,* $At_2 = \{q_1, q_2\}$*,* $K = \{x, y, z\}$*, and* $\preceq_i$ *are arbitrary. The outcome function is represented by the following matrix.*

	$\neg q_1 \wedge \neg q_2$	$q_1 \wedge \neg q_2$	$\neg q_1 \wedge q_2$	$q_1 \wedge q_2$
$\neg p_1 \wedge \neg p_2$	x	x	z	z
$p_1 \wedge \neg p_2$	z	x	z	y
$\neg p_1 \wedge p_2$	z	z	y	y
$p_1 \wedge p_2$	z	z	x	z

We can in fact define the atoms in K in terms of propositional formulae made of atoms in At. For instance $x \leftrightarrow (\neg q_1 \wedge \neg q_2 \wedge \neg p_1 \wedge \neg p_2) \vee (q_1 \wedge \neg q_2 \wedge \neg p_1 \wedge \neg p_2) \vee (q_1 \wedge \neg q_2 \wedge p_1 \wedge \neg p_2) \vee (\neg q_1 \wedge q_2 \wedge p_1 \wedge p_2)$.

We could have used a preference relation over the set of valuations Θ, rather than over the set of outcomes K. However, this would not sit compatibly with the notion of a strategic game. Such a semantical choice commits the designer to take into account, not only the outcomes of a game, but also the means of bringing about a result. This could be achieved in our models by imposing that the outcome function o is a bijection, but this is not the case in general. Note that, in terms of compactness, making use of a set of consequences is interesting when its size is very small wrt. the set of valuations. This is however a tangential issue and in this article we aim for generality and ease of modelling.

Definition 8 (truth values of $\mathcal{L}(N, At, K)$). *The truth value of a formula of $\mathcal{L}(N, At, K)$ is wrt. a GPCC G and an N-valuation θ. It is inductively given by:*

$$
\begin{array}{lll}
G, \theta \models \top & & \\
G, \theta \models x & \textit{iff} & o(\theta) = x \qquad , x \in K \\
G, \theta \models p & \textit{iff} & \theta(p) = \mathtt{tt} \qquad , p \in At \\
G, \theta \models \neg\varphi & \textit{iff} & G, \theta \not\models \varphi \\
G, \theta \models \varphi \vee \psi & \textit{iff} & G, \theta \models \varphi \textit{ or } G, \theta \models \psi \\
G, \theta \models \Diamond_C \varphi & \textit{iff} & \textit{there is a } \theta' \in \Theta \textit{ such that } \theta' \equiv_{N \setminus C} \theta \textit{ and } G, \theta' \models \varphi \\
G, \theta \models \langle \preceq_i \rangle \varphi & \textit{iff} & \textit{there is a } \theta' \in \Theta \textit{ such that } o(\theta) \preceq_i o(\theta') \textit{ and } G, \theta' \models \varphi
\end{array}
$$

The truth of φ in all models is denoted by $\models \varphi$. The classical operators $\wedge$, $\rightarrow$, $\leftrightarrow$ can be defined as usual. We also define $\Box_C \varphi \triangleq \neg \Diamond_C \neg \varphi$ and $[\preceq_i]\varphi \triangleq \neg\langle \preceq_i \rangle \neg \varphi$. Observe that $\Box_N$ plays the role of a universal modality, that is, it allows to quantify over all the N-valuations. We will elaborate on the modal operators in our language below.

In order to properly link this to a modal logic approach (which we will exploit in our completeness proof), we need one more definition.

Definition 9 (Kripke models with control and consequences). *A* Kripke model with propositional control and consequences *(notation: MPCC) is a tuple $M = \langle \Theta, N, K, o, At, At_{i\in N}, R_{i \in N}, P_{i \in N} \rangle$ where Θ, N, K, o, At and $At_{i \in N}$ are as in GPCC's, and R_i and P_i are defined as follows. $R_i \theta_1 \theta_2$ iff $\theta_1 \equiv_{N \setminus \{i\}} \theta_2$. For coalitions C, we define $R_C = \circ_{i \in C} R_i$ (where $R_i \circ R_j = \{(s,t) \mid \exists u : R_i su \,\&\, R_j ut\}$). Finally, P_i is a preference relation over K, for every $i \in N$. We lift this to a preference relation $\sqsubseteq_i$ as follows: $\theta_1 \sqsubseteq_i \theta_2$ iff $o(\theta_1) \preceq_i o(\theta_2)$. The truth definition of formulae is in line with that of Definition 8, but now $\Diamond_C$ is a diamond operator wrt. to R_C, and $\langle \preceq_i \rangle \varphi$ is a diamond operator with respect to $\sqsubseteq_i$. For technical reasons (i.e., for our completeness proof), it is sometimes convenient to represent an MPCC as a model with an abstract set of states S, rather than directly valuations. We say that $M_S = \langle S, N, K, o', At, At_{i \in N}, R'_{i \in N}, P_{i \in N}, V \rangle$, where V is a valuation $V : S \longrightarrow (At \longrightarrow \{\mathtt{tt}, \mathtt{ff}\})$,* simulates *$M = \langle \Theta, N, K, o, At, At_{i \in N}, R_{i \in N}, P_{i \in N} \rangle$, if there is a bijection $F : S \longrightarrow \Theta$ such that for all $s, s_1, s_2 \in S$ and all $p \in At$, we have $F(s)(p) = V(s)(p)$, $o(F(s)) = o'(s)$ and $R_i F(s_1) F(s_2)$ iff $R'_i s_1 s_2$.*

Rather than having outcome functions in M we might as well have used valuations over $At \cup K$, with some additional constraints on them. Note that $\theta \in \Theta$ in G is conceived as an action profile, whereas in a model M it is a state which is completely determined by its valuation to At. The following is easily verified.

Observation 1. *With every GPCC G we can identify a MPCC M_G such that for all θ and φ, we have $G, \theta \models \varphi$ iff $M_G, \theta \models \varphi$, and, conversely, for every MPCC M there is a GPCC G_M such that for all θ and φ, we have $M, \theta \models \varphi$ iff $G_M, \theta \models \varphi$. Moreover, if M_S simulates M, then for all s and φ: $M, F(s) \models \varphi$ iff $M_S, s \models \varphi$.*

If one accepts a treatment of social choice in modal logic, our language is simple and natural. It only requires extending classical logic with a basic action (local ability) operator for every coalition and a standard modal preference operator for every individual agent. Now we might want a more elaborate vocabulary that could be needed to talk about game properties.

Local effectivity, represented by $\Diamond_C \varphi$, holds in a strategy profile θ if the agents in C could deviate from their current action such that φ holds in the resulting profile. For instance, if $p_i \in At_i$ and in the current situation θ all atoms are false, then $\Diamond_{\{1,2\}}(p_1 \wedge \neg(p_2 \vee p_3))$ holds (i.e., 1 can decide to toggle p_1 to make it true, 2 may decide not to make p_2 true and since $3 \notin C$, it is assumed he does not touch his variables. $\Diamond_C \varphi$ is the actional primitive of our logic. Note that $\Box_C \varphi$ in θ then means that given θ, no matter how the agents in C would change their choice, φ holds. The notion of *brute choice*, or actual agency, is strongly related to the one of local effectivity. Chellas' stit operator [15] is the archetype of operator of choice: $[C]\varphi$ reads "the coalition C choose such that φ", or "the current choice of players in C being equal, φ holds whatever other agents do". We can define it as follows:

$$[C]\varphi \triangleq \Box_{N \setminus C} \varphi.$$

Then, $[C]\varphi$ in θ expresses that the complement of C cannot locally bring about $\neg\varphi$. Its dual $\langle C \rangle \varphi \triangleq \neg[C]\neg\varphi$ reads "coalition C allow φ", or "the current choice of players in C does not rule out that φ".

We can also define a useful vocabulary for preferences. The formula $\langle \preceq_i \rangle \varphi$ holds in θ if there is a profile θ' for which φ is true, and of which the outcome is preferred over the current one. We can start by defining the strict preference counterpart of $\langle \preceq_i \rangle$ as follows:

$$\langle \prec_i \rangle \varphi \triangleq \bigvee_{\theta \in \Theta} (\pi(\theta) \wedge \langle \preceq_i \rangle (\varphi \wedge \neg \langle \preceq_i \rangle \pi(\theta)).$$

Note that for every θ_N, $\pi(\theta_N)$ is true at one and only one state. The reification $\pi(\theta_N)$ somewhat plays the role of a nominal in Hybrid Logic [6]. This is the reason why we are able to give such a definition, while asymmetry is undefinable in modal logic.

Another operator of interest is $\psi \preceq^i_{\forall\forall} \varphi$, corresponding to a weak preference between propositions. It reads "all φ are better than all ψ".

$$\psi \preceq^i_{\forall\forall} \varphi \triangleq \Box_N \bigvee_{\theta \in \Theta} (\pi(\theta) \wedge (\varphi \rightarrow \Box_N(\psi \rightarrow \langle \preceq_i \rangle \pi(\theta)))).$$

Some variants are also easy to define. For instance, von Wright's definition of *ceteris paribus* preferences corresponds to the strict counterpart of $\preceq^i_{\forall\forall}$, and can simply be simulated by substituting the preference modality of the definition by the strict one defined above. Moreover, substituting the $\Box_N$ modalities for $\Diamond_N$ allows to capture related yet different notions of preferences. For instance, we can grasp a preference operator that reads "there is a φ that is better than all ψ" by substituting the first operator $\Box_N$ of the definition with $\Diamond_N$. See [23, 12] for the details on the logic of these operators.

2.2 Principles

In this section, we give the principles that axiomatise the class of models defined before. We establish that the resulting logical system determines the class of GPCC.

CL-PC	
(*Prop*)	φ , where φ is a propositional tautology
(*K*(*i*))	$\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$
(*T*(*i*))	$\Box_i\varphi \to \varphi$
(*B*(*i*))	$\varphi \to \Box_i\Diamond_i\varphi$
(*comp*∪)	$\Box_{C_1}\Box_{C_2}\varphi \leftrightarrow \Box_{C_1\cup C_2}\varphi$
(*empty*)	$\Box_\emptyset\varphi \leftrightarrow \varphi$
(*exclu*)	$(\Diamond_i p \wedge \Diamond_i\neg p) \to (\Box_j p \vee \Box_j\neg p)$, where $j \neq i$
(*actual*)	$\bigvee_{i\in N} \Diamond_i p \wedge \Diamond_i\neg p$
(*full*)	$(\bigwedge_{p\in X} \Diamond_i p \wedge \Diamond_i\neg p) \to \Diamond_i\varphi_v$, where φ_v is the conjunction of literals true in one valuation v of $X \subseteq At$
Outcomes and preferences	
(*func*1)	$\bigvee_{x\in K}(x \wedge \bigwedge_{y\in K\setminus\{x\}} \neg y)$
(*func*2)	$(\pi(\theta) \wedge x) \to \Box_N(\pi(\theta) \to x)$
(*incl*)	$\Box_N\varphi \to [\preceq_i]\varphi$
(*K*(⪯ᵢ))	$[\preceq_i](\varphi \to \psi) \to ([\preceq_i]\varphi \to [\preceq_i]\psi)$
(4(⪯ᵢ))	$\langle\preceq_i\rangle\langle\preceq_i\rangle\varphi \to \langle\preceq_i\rangle\varphi$
(*connect*)	$(\varphi \wedge \Diamond_N\psi) \to \langle\preceq_i\rangle\psi \vee \Diamond_N(\psi \wedge \langle\preceq_i\rangle\varphi)$
(*unifPref*)	$(x \wedge \langle\preceq_i\rangle y) \to x \preceq^i_{\forall\forall} y$
Rules	
(*MP*)	from $\vdash_\Lambda \varphi \to \psi$ and $\vdash_\Lambda \varphi$ infer $\vdash_\Lambda \psi$
(*Nec*(□ᵢ))	from $\vdash_\Lambda \varphi$ infer $\vdash_\Lambda \Box_i\varphi$

Figure 1: Axiomatics of $\Lambda(N, At, K)$. i is over N, C_1 and C_2 over 2^N, φ represents an arbitrary formula of $\mathcal{L}(N, At, K)$, p is over At, x and y are over K, θ is over Θ (the set of valuations of At).

The system $\Lambda(N, At, K)$ is presented in Figure 1. The upper part corresponds to the fragment of CL-PC. (We essentially used the axiomatics from [10].) (*comp*∪) defines the ability of coalitions in terms of abilities of sub-coalitions. (*empty*) means that the empty coalition has no power. (*exclu*) means that an atom of At is controlled by at most one agent. (*actual*) means that an atom is controlled by at least one agent. (*full*) means that an agent i is able to play every possible valuation of a set of atoms controlled by i. The second part gives the principles associated to the outcomes and preferences. (*incl*) ensures that if something is settled, a player cannot prefer its negation. Reflexivity of preferences is a consequence of totality, ensured by (*connect*). (4($\preceq_i$)) characterises transitivity. (*func*1) forces the fact that for every action profile there is one and only one outcome. (*func*2) ensures that outcomes are only determined by the valuations. (*unifPref*) specifies a fundamental interaction between preferences and the outcomes. If the action profile at hand leads to x and agent i prefers an action profile leading to y, then at every action profile leading to x, agent i will prefer every action profile leading to y, that is, all y are better than all x. The presence of ($K(i)$) with ($Nec(\Box_i)$) gives to the operator $\Box_i$ the property of normality. The normality of the modality $[\preceq_i]$ follows because of ($K(\preceq_i)$) and (*incl*).

The following lemma is given without proof . It is instrumental in the proof of Theorem 1.

Lemma 1. *The following are derivable.*

1. $(\Diamond_i p \wedge \Diamond_i\neg p) \to \Box_N(\Diamond_i p \wedge \Diamond_i\neg p)$;

2. $\Diamond_N(\pi(\theta) \wedge \psi) \to \Box_N(\pi(\theta) \to \psi)$;

3. $\Diamond_N(\psi \wedge y) \to (\Box_N(x \to \langle\preceq_i\rangle y) \to \Box_N(x \to \langle\preceq_i\rangle\psi))$.

The next result states that $\Lambda(N, At, K)$ is determined by the class of games of propositional control with consequences.

Theorem 1. *The system $\Lambda(N, At, K)$ is sound and complete with respect to the class GPCC of games of propositional control with consequences.*

Proof. Soundness is routine. In order to prove completeness, assume that φ is consistent in $\Lambda(N, At, K)$. We will show that there is a M_S with a state s for which $M_S, s \models \varphi$, and moreover, M_S simulates an MPCC M. The claim then follows from Observation 1. We adapt a standard modal logic technique ([8]).

Let Ξ be the set of all maximal consistent (mc.) sets in $\Lambda(N, At, K)$. For $\Gamma, \Delta \in \Xi$, define $R^\Xi_i \subseteq \Xi \times \Xi$ by $R^\Xi_i\Gamma\Delta$ iff for all $\delta \in \Delta, \Diamond_i\delta \in \Gamma$. Let R^Ξ be the transitive closure of $\cup_{i\in N}R^\Xi_i$. Starting with a consistent formula φ, we know there is an mc. Γ_φ with $\varphi \in \Gamma_\varphi$. We now define the model $M_\varphi = \langle S, N, K, o', At, At_{i\in N}, R'_{i\in N}, P_{i\in N}, V\rangle$ as follows. The sets N, At and K are parameters of the language, and given. Take for S the set $\{\Delta \in \Xi \mid R^\Xi\Gamma_\varphi\Delta\}$. Define $o'(\Delta) = x$ iff $x \in \Delta$. Let $At_i = \{p \in At \mid \Diamond_i p \wedge \Diamond_i\neg p \in \Gamma_\varphi\}$. The relations R'_i are the restrictions of R^Ξ_i to S, and P'_ixy holds iff $\Diamond_N(x \wedge \langle\preceq_i\rangle y) \in \Gamma_\varphi$. Finally, put $V(\Delta)(p) = p \in \Delta$.

Claim 1. *For all $\Delta \in S, \delta \in \mathcal{L}(N, At, K)$: $M_\varphi, \Delta \models \delta$ iff $\delta \in \Delta$.*

We only prove the $\langle\preceq_i\rangle$ case. So suppose $\langle\preceq_i\rangle\psi \in \Delta$. Then, by (*Prop*) and (*func*1), there must be some $\theta \in \Theta$ and $y \in K$ for which $\langle\preceq_i\rangle(\psi \wedge \pi(\theta) \wedge y) \in \Delta$. By (*func*1), we know there is a unique outcome $x \in \Delta$. So, we have $x \wedge \langle\preceq_i\rangle(\psi \wedge \pi(\theta) \wedge y) \in \Delta$ and hence $(x \wedge \langle\preceq_i\rangle y) \in \Gamma_\varphi$. This means there is some $\Psi \in S$ for which $y, \psi \in \Psi$. By induction, we have $M_\varphi, \Psi \models \psi$, and by definition of P_i we then have $M_\varphi, \Delta \models \langle\preceq_i\rangle\psi$. Conversely, suppose $M_\varphi, \Delta \models \langle\preceq_i\rangle\psi$. Assuming the outcome for Δ to be x, there is some $\Psi \in S$ and a $y \in K$ and $\theta \in \Theta$ such $M_\varphi, \Psi \models (\psi \wedge y \wedge \pi(\theta))$ and P_ixy. By induction and the construction of S, we have $\Diamond_N(\psi \wedge y \wedge \pi(\theta)) \in \Gamma_\varphi$ and also we have $\Box_N(x \to \langle\preceq_i\rangle y) \in \Gamma_\varphi$. By Lemma 1, item 3, we then have $\Box_N(x \to \langle\preceq_i\rangle\psi) \in \Gamma_\varphi$, and hence $\langle\preceq_i\rangle\psi \in \Delta$.

Claim 2. *The model M_φ simulates an MPCC M.*

To prove this claim, we first argue that for for every $\theta \in \Theta$, there is exactly one $\Delta \in S$ such that $\pi(\theta) \in \Delta$. So let $\theta \in \Theta$. Consider Γ_φ. By (*actual*), for every atom p, there is some agent $i \in N$ for which $(\Diamond_i p \wedge \Diamond_i\neg p) \in \Gamma_\varphi$. By (*full*) and (*comp*∪), we find that $\Diamond_N\pi(\theta) \in \Gamma_\varphi$, and hence there must be some mc. set Δ with $\pi(\theta) \in \Delta$ and $\Delta \in S$. Now suppose $\Delta' \in S$ also contains $\pi(\theta)$. By Lemma 1 item 2, we conclude that Δ and Δ' must contain the same formulae, and hence they are equal.

Then, we take $F : S \longrightarrow \Theta$ as follows: $F(\Delta) = \theta$ iff $\pi(\theta) \in \Delta$. It is easy to see that $F(\Delta)(p) = V(s)(p)$ and we can define $o(F(\Delta)) = o'(\Delta)$. Let $F(\Delta_1) = \theta_1$ and $F(\Delta_2) = \theta_2$. The proof that $R_i\theta_1\theta_2$ iff $R'_i\Delta_1\Delta_2$ is omitted. ∎

3. SOCIAL CHOICE AND SOLUTION CONCEPTS

In this section, we show how the logic can be applied to some questions in social choice theory. We first study the powers of coalitions in game of propositional control with consequences in K. Then we formalise various notions of game equilibria.

3.1 Effectivity of coalitions

A game form is said to be rectangular if the inverse image of any outcome is a Cartesian product. More formally, for every $x \in K$, there is a subset V_i of the set of i-valuations for every player i, such that we have:

$$o^{-1}(x) = \prod_{i\in N} V_i.$$

It is easy to see that CL-PC represents special rectangular game forms such that the inverse of an outcome is the Cartesian product of singletons. This implies that it models a particular type of game forms where the outcome function is a bijection between the set of action profiles and the set of consequences.

Rectangular games have interesting properties, but there is no *a priori* reason to exclude non-rectangular games. Problems in SCT are usually about whether coalitions are effective for some subset of outcomes $X \subseteq K$, and identifying the outcomes of a game with its set of action profiles is a limitation. In this section, we propose to show how much different the models of CL-PC are from their new extension GPCC in terms of effectivity modelling.

Example 2. *Consider the following game form:*

	p_2	$\neg p_2$
p_1	x	x
$\neg p_1$	y	y

Clearly, it is rectangular, however, there is no bijection between Θ and K: $o^{-1}(x) = \{p_1\} \times \{p_2, \neg p_2\}$ and $o^{-1}(y) = \{p_2\} \times \{p_2, \neg p_2\}$.

But more interestingly, models of our logic do not necessarily represent rectangular games at all.

Example 3. *Consider the following game form where player 1 chooses rows, player 2 chooses columns and player 3 chooses matrices: for every player i we define $At_i = \{p_i\}$.*

p_3:

	p_2	$\neg p_2$
p_1	x	x
$\neg p_1$	y	z

$\neg p_3$:

	p_2	$\neg p_2$
p_1	x	z
$\neg p_1$	y	z

Clearly $o^{-1}(y) = \{\neg p_1\} \times \{p_2\} \times \{p_3, \neg p_3\}$, but neither $o^{-1}(x)$ nor $o^{-1}(z)$ is a Cartesian product.

If GPCC does not represent rectangular games, it is then interesting to investigate what are actually the powers of the coalitions underlying GPCC, and see what differs with CL-PC. We use *effectivity functions* (EF) which are a general tool for this purpose [2].

Some definitions are in order. An effectivity function is a mapping E that associates a collection of subsets of K to every coalition C. $E(C)$ represents the sets of outcomes for which the coalition C is 'effective'. The notion of effectivity is a very abstract one and really depends on the application that one has in mind. Some minimal requirements are nevertheless acknowledged:

Definition 10. *An* effectivity function *is a function $E : 2^N \longrightarrow 2^{2^K}$ such that (i) $\emptyset \notin E(C)$; (ii) $K \in E(C)$; (iii) $E(\emptyset) = \{K\}$; (iv) $E(N) = 2^K \setminus \{\emptyset\}$.*

We now define significant properties of EF. For clarity, we define them in two bundles.

Definition 11 (standard EF). *An effectivity function E is said to be standard if we have the following properties:*

- outcome monotony*: $\forall C \in 2^N, \forall A, A' \in 2^K$ if $A \in E(C)$ and $A \subseteq A'$ then $A' \in E(C)$;*
- agent monotony*: $\forall C, C' \in 2^N, \forall A \in 2^K$, if $A \in E(C)$ and $C \subseteq C'$ then $A \in E(C')$;*
- superadditivity*: $\forall C, C' \in 2^N, \forall B, B' \in 2^K$, if $C \cap C' = \emptyset$, $B \in E(C)$ and $B' \in E(C')$ then $B \cap B' \in E(C \cup C')$.*

As we shall see later, standard effectivity functions naturally rise in game theory. There are at least two other properties that are relevant in this quick study. First, we recall that the *nonmonotonic core* of an effectivity function E is defined as $\mu(E, C) = \{B \in E(C) \mid \forall B' \subset B, B' \notin E(C)\}$. The nonmonotonic core of an effectivity function E for a coalition C is the collection of minimal sets for inclusion in $E(C)$.

Definition 12 (other EF properties). *An effectivity function E is said to be:*

- converse superadditive *iff $\forall C, C' \in 2^N, \forall B \in 2^K$ if $C \cap C' = \emptyset$ and $B \in E(C \cup C')$ then $\exists A \in E(C)$ and $\exists A' \in E(C')$ such that $B = A \cap A'$;*
- decisive *iff for any $(B_1, \ldots, B_n) \in \mu(E, \{1\}) \times \ldots \times \mu(E, \{n\})$ we have $Card(B_1 \cap \ldots \cap B_n) = 1$.*

Effectivity functions are simple tools for describing the space of strategies in some interaction scenari. They are particularly useful for encoding the powers in a game. We can come with several notions of effectivity in a game. Perhaps the most useful is the *alpha effectivity* which corresponds to a 'pessimistic' type of behaviour of the coalitions: a coalition is said to be alpha effective for $X \subseteq K$ if it can force the solution to be in X whatever other agents do.[1]

Remark 1. *$\Diamond_C[C]\varphi$ is true for every property φ for which a coalition C is effective. It does not depend on a particular N-valuation of evaluation and is globally true (or false) in a GPCC: $\vdash_\Lambda \Diamond_C[C]\varphi \leftrightarrow \Box_N\Diamond_C[C]\varphi$.*

We define formally the alpha effectivity E^G_α of a GPCC G as follows:

$$E^G_\alpha(C) = \left\{ X \subseteq K \mid G \models \Diamond_C[C] \bigvee_{x \in X} x \right\}.$$

Note that alpha effectivity does not depend on preferences. The alpha effectivity of a game is the effectivity of its underlying mechanism, whatever the preferences are.

We can now formulate a theorem proved in [1] that can help us at understanding the structure of powers in the games of propositional control.

Theorem 2 ([1]). *An effectivity function E is the alpha effectivity function of some rectangular game form iff E is standard, decisive and conversely superadditive.*

Since we have shown that some games represented by a GPCC are not rectangular, at least one of the properties of effectivity function that we defined above must not hold. Again, it demonstrates that GPCC generalises the notion of effectivity of CL-PC. The next proposition states that the alpha effectivity function of a GPCC verifies the properties of standard effectivity functions.

Proposition 1. *For every game of propositional control with consequences in K, its alpha effectivity function is standard.*

Proof. It is easy to check that $E^G_\alpha(\emptyset) = \{K\}$ and $E^G_\alpha(N) = 2^K \setminus \{\emptyset\}$. Outcome monotony is a consequence of modal logic, agent monotony follows from $(comp\cup)$.

The case of superadditivity is slightly more involved. In [25], the schema (S') is proved: $\vdash_\Lambda \Diamond_C\varphi \wedge \Diamond_{C'}\psi \rightarrow \Diamond_{C\cup C'}(\varphi \wedge \psi)$ (where φ and ψ do not contain any common atom of At). Suppose $C \cap C' = \emptyset$, $X \in E^G_\alpha(C)$ and $X' \in E^G_\alpha(C')$. Then $G \models \Diamond_C[C] \bigvee_{x\in X} x$ and $G \models \Diamond_{C'}[C'] \bigvee_{y \in X'} y$. Since these formulae do not contain any atom of At, (S') implies that $G \models \Diamond_{C\cup C'}([C] \bigvee_{x\in X} x \wedge [C'] \bigvee_{y\in X'} y)$. Equivalently, $G \models \Diamond_{C\cup C'}(\Box_{N\setminus C} \bigvee_{x\in X} x \wedge \Box_{N\setminus C'} \bigvee_{y\in X'} y)$. Observe that because by hypothesis $C \cap C' = \emptyset$, we have $N\setminus C = C' \cup (N \setminus (C \cup C')$. It means that $\Box_{N\setminus C}\varphi \leftrightarrow \Box_{C'}\Box_{N\setminus\{C\cup C'\}}\varphi$. Thus $\Box_{N\setminus C}\varphi \leftrightarrow \Box_{C'}[C \cup C']\varphi$ and symmetrically $\Box_{N\setminus C'}\varphi \leftrightarrow \Box_C[C \cup C']\varphi$. Using standard modal logic we conclude that $G \models \Diamond_{C\cup C'}[C \cup C'] \bigvee_{z \in X\cap X'} z$. ■

[1]On the 'optimistic' side is also the *beta effectivity*: a coalition C is said beta effective for $X \subseteq K$ if for every combined action of the other players, C can contingently ensure the solution to be in X.

This is not at all surprising. Standard effectivity functions form an important class: an effectivity function E is standard iff it is the alpha effectivity function of some strategic game form.

The next two propositions state that the alpha effectivity function of a GPCC is not conversely superadditive and not decisive in general.

Proposition 2. *For some GPCC, the alpha effectivity function is not conversely superadditive.*

Proof. Consider the game form in Example 3. We have $E^M_\alpha(\{1\}) = \{\{x, y\}, \{y, z\}, \{x, y, z\}\}$, $E^M_\alpha(\{2\}) = \{\{x, y\}, \{x, z\}, \{x, y, z\}\}$, $E^M_\alpha(\{3\}) = \{\{x, y, z\}\}$, $E^M_\alpha(\{1, 2\}) = 2^K \setminus \{\emptyset\}$, $E^M_\alpha(\{1, 3\}) = \{\{x\}, \{y, z\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$, $E^M_\alpha(\{2, 3\}) = \{\{z\}, \{x, y\}, \{x, z\}, \{x, y, z\}\}$.

We have $\{1\} \cap \{3\} = \emptyset$, $\{x\} \in E^M_\alpha(\{1, 3\})$ but there are no $X \in E^M_\alpha(\{1\})$ and $X' \in E^M_\alpha(\{3\})$ such that $X \cap X' = \{x\}$. ■

It is also very easy to see that decisiveness will not hold in general, and the game of Example 3 is a counterexample.

Proposition 3. *For some games of propositional control with consequences in K, the alpha effectivity function is not decisive.*

In summary, the addition of a set of consequences K leads to a weaker notion of effectivity than CL-PC. In general, the logic models standard effectivity functions. We thus have a logic capable of the main features of Coalition Logic. However, it is not weaker than Coalition Logic in this respect. Note that we did not wholly prove the characterisations of the effectivities underlying CL-PC or GPCC. Such results can be adapted from [9] and [19] respectively.

What is remarkable is that it also combines well with preferences and allows us to encode a wide range of solution concepts. This is the aim of the next section.

3.2 Solution concepts

We define some important solution concepts in *pure strategies*. Those are definitions of very standard notions of game theory, viz. weak Pareto optimality, very weak dominance, Nash equilibrium, strong Nash equilibrium and the concept of weak core.

We recall for convenience the interpretations of the operators that we consider as central in the formalisation of most solution concepts. $\Box_C\varphi$ reads "for every deviation of the players in C, φ is true", and $[C]\varphi$ reads "the current action of C ensures φ whatever other agents do". They are inter-definable: $\Box_C\varphi \leftrightarrow [N \setminus C]\varphi$.

In the following definitions, we assume $\langle N, At, (At_i), K, o, (\preceq_i)\rangle$, a GPCC as defined in Definition 7.

Definition 13 (Pareto optimality). *An N-valuation θ^* is a* weak Pareto optimum *iff there is no N-valuation θ such that $o(\theta)$ is strictly preferred over $o(\theta^*)$ by every agent.*

$$WPO \triangleq \bigvee_{x\in K}\left(x \wedge \Box_N \bigvee_{i\in N}\langle\preceq_i\rangle x\right).$$

The action profile at hand is weakly Pareto optimal iff, x is the outcome and at every action profile there is an agent that would prefer x. The definition of Pareto optimality does not rely on the space of strategies of the players, but essentially on the outcomes. It is straightforward that if a valuation (or action profile in GT) is weakly Pareto optimal then every valuation whose outcome is x is a weak Pareto optimum: $\vdash_\Lambda (x \wedge WPO) \to \Box_N(x \to WPO)$. Clearly we do not need the full expressivity of our language since a universal modality suffices. The remaining solution concepts of this section will have a more contingent flavour.

Definition 14 (dominance equilibria). *θ^* is a* very weakly dominant *N-valuation iff for every player i and $N \setminus \{i\}$-valuation θ_{-i}, i considers $o(\theta^*_i, \theta_{-i})$ at least as good as $o(\theta'_i, \theta_{-i})$ for every i-valuation θ'_i.*

It is handy to introduce the notion of weak best response by an agent i.

$$WBR_i \triangleq \bigvee_{x\in K}(x \wedge \Box_i\langle\preceq_i\rangle x).$$

A player i plays a weak best response in an N-valuation if, x being the outcome, for every deviation of i, i prefers x.

We can now define very weak strategy dominance in terms of weak best response:

$$VWSD \triangleq \bigwedge_{i\in N}[i]WBR_i.$$

We have a very weak strategy dominant valuation if the current choice of every player ensures her a weak best response whatever other agents do.

Definition 15 (Nash equilibrium). *An N-valuation θ^* is a* Nash equilibrium *iff for every player $i \in N$ and for all i-valuations θ_i, i considers $o(\theta^*_i, \theta^*_{-i})$ at least as good as $o(\theta_i, \theta^*_{-i})$.*

$$NE \triangleq \bigwedge_{i\in N} WBR_i.$$

A valuation is a Nash equilibrium if every player plays a weak best response. Interestingly, [24] proposed a similar definition along the pattern $\bigwedge_{i\in N}[N \setminus \{i\}]\langle\preceq_i\rangle x$ within an epistemic language and where $[N \setminus \{i\}]$ is intended to represent the distributed knowledge in $N\setminus\{i\}$.

To complete this collection of solution concepts, we can also show relevance of GPCC for cooperative games via the study of strong Nash equilibrium and the core of strategic games.

Definition 16 (strong Nash equilibrium). *An N-valuation θ^* is a* strong Nash equilibrium *iff there is no coalition $C \subset N$ and no C-valuation θ_C such that $o(\theta_C, \theta^*_{-C})$ is considered strictly better than $o(\theta^*)$ by every players of C.*

$$SNE \triangleq \bigvee_{x\in K}\left(x \wedge \bigwedge_{C\subset N} \Box_C \bigvee_{i\in C}\langle\preceq_i\rangle x\right).$$

A players' valuation is a strong Nash equilibrium if, the outcome being x, for every deviation of C one of its member prefers x.

Definition 17 ((weak) core). *An N-valuation θ^* is* dominated *iff there is a coalition $C \subset N$ and a C-valuation θ_C such that for all $N \setminus C$-valuation θ_{-C}, every $i \in C$ strictly prefers $o(\theta_C, \theta_{-C})$ over $o(\theta^*)$. θ^* is in the (weak)* core *iff it is not dominated.*

This last definition holds for a *coalitional game without transferable utilities*. That is, players can form coalitions, but cannot redistribute the sum of the payoffs among the individuals of the coalition.

$$INCR \triangleq \bigvee_{x\in K}\left(x \wedge \bigwedge_{C\subset N} \Box_C\langle C\rangle \bigvee_{i\in C}\langle\preceq_i\rangle x\right).$$

An N-valuation is in the weak core if, the outcome being x, for every deviation of C, C allows that one of its member prefers x.

We state the following without proof.

Proposition 4. *Assume a GPCC $G = \langle N, At, (At_i), K, o, (\preceq_i)\rangle$ and an N-valuation θ^*. We have that θ^* is:*

- *weak Pareto optimal iff $G, \theta^* \models WPO$;*
- *very weak strategy dominant iff $G, \theta^* \models VWSD$;*
- *Nash equilibrium iff $G, \theta^* \models NE$;*
- *strict Nash equilibrium iff $G, \theta^* \models SNE$;*
- *in the (weak) core iff $G, \theta^* \models INCR$.*

4. EXAMPLE APPLICATIONS

We present two applications of our logic, one being an instance of model checking and the other an application of theorem deduction.

4.1 Finding equilibria

Typically in game logics, the characterisation of solution concepts is achieved by defining predicates of the form $SC((a_i)_{i\in N})$, stating that the particular action profile $(a_i)_{i\in N}$ is an equilibrium with respect to the solution concept SC.

For model checking solution concepts we would like to give as input (i) a game, and (ii) a *general* formulation of a solution concept, and obtain as output the set of action profiles that verify it. A general problem of model checking can be formally stated as follows: given a logical formula φ and a model $\mathcal{M}$, return the set of states S such that $s \in S$ iff $\mathcal{M}, s \models \varphi$. Model checking approaches to verifying solution concepts are somewhat limited in existing logics because the modeller first has to choose an action profile $(a_i)_{i\in N}$ and then check whether the game satisfies $SC((a_i)_{i\in N})$.

In Section 3.2, we have been able to give a general logical formulation SC for some important solution concepts in strategic games. As a consequence, we can check in a very natural manner where are the equilibria of a game.

Given $G = \langle N, At, (At_i), K, o, (\preceq_i)\rangle$ a GPCC, and a solution concept SC, model checking SC against G amounts at determining the set $\{\theta \in \Theta \mid G, \theta \models SC\}$.

As we suggested before, an N-valuation in GPCC can be seen as a nominal in Hybrid Logic. In [21], we develop this idea in the framework of Hybrid Logic which appears particularly convenient for model checking this problem.

4.2 Game form solvability

Powers of coalitions and solution concepts meet naturally in the literature on game form solvability. The problem of game form solvability in terms of GPCC is defined as follows:

Definition 18. *Let a solution concept SC be given. A game form $M = \langle N, At, (At_i), K, o\rangle$ is said to be SC-*solvable *iff for every preference profile $(\preceq_i)$ over K the GPCC $\langle M, (\preceq_i)\rangle$ admits an SC equilibrium.*

Nash equilibrium is much celebrated in game theory, and it is also a widely studied notion of game form solvability. Some sufficient conditions have been given in literature. The most salient result is the one of [20] where the authors give a sufficient and necessary condition on the structure of an effectivity function E for a game representing E to admit a Nash equilibrium. [13] capitalises on this result in order to build an extension of Coalition Logic that conceptualises the class of Nash solvable representations of effectivity functions. To achieve this, it is sufficient to add a single inference rule and does not require a more expressive object language than in basic CL.

In social choice theory, it is certainly elegant to force Nash solvability only by constraining further the powers of coalitions. However, we take here a different and more practical stance. We use the fact that there is a formula in our language that is true iff it is evaluated against an N-valuation that is a Nash equilibrium (Proposition 4). It appears that our logic makes the task of reasoning about Nash solvable mechanisms particularly flexible. This will be true in fact for every solution concept that we are able to define in GPCC. Here, we focus on Nash solvability.

As an illustration we can study two examples of [11]: a version of the so-called Gibbard's paradox, and the wedding scenario.

Example 4. *Two individuals a and e both have a blue and a white shirt. Each agent has the right to choose the colour of its own shirt. In formula, if p_i means that player i wears its white shirt, the rights of the systems are simply given as follows:*

$$\rho \triangleq \Diamond_a p_a \wedge \Diamond_a \neg p_a \wedge \Diamond_e p_e \wedge \Diamond_e \neg p_e.$$

The 'paradox' tells us that there is no Nash solvable implementation of this scenario. In other words $\rho \to \Diamond_N NE$ is not a valid sentence of the logic $\Lambda(\{a, e\}, \{p_a, p_e\}, \{x_1, x_2, x_3, x_4\})$.[2] *For example, we can build a counter model if the agents' preferences are such that: the outcome function is bijective, a prefers white over blue and prefers to wear the same colour as b; and b also prefers white over blue but prefers wearing a different colour than the one worn by a.*

The second example is, on the other hand, a case where the scenario does have a Nash solvable representation.

Example 5. *There are three players: Angelina (a), Edwin (e) and the male Judge (j). Angelina can choose to marry Edwin or the Judge or stay single. Edwin and the Judge can choose to stay single or to marry Angelina. The consequences are that Angelina marries Edwin (m_e) (resp. the Judge (m_j)) when both of them choose so, or she stays single (s).*

We can model this scenario using a model such that $N = \{a, e, j\}$, $K = \{s, m_e, m_j\}$ $At_a = \{p_a, p'_a\}$, $At_j = \{p_j\}$ and $At_e = \{p_e\}$. The controls of the scenario are determined by the formula:

$$\rho \triangleq \Diamond_a p_a \wedge \Diamond_a \neg p_a \wedge \Diamond_a p'_a \wedge \Diamond_a \neg p'_a \wedge \Diamond_e p_e \wedge \Diamond_e \neg p_e \wedge \Diamond_j p_j \wedge \Diamond_j \neg p_j.$$

p'_a means that Angelina decides not *to stay single. When p'_a is true, p_a means that she chooses to marry Edwin and $\neg p_a$ means that she chooses to marry the Judge. p_e (resp. p_j) means that Edwin (resp. the Judge) decides to marry Angelina. Obviously, we have:*

$$\omega_{m_e} \triangleq m_e \leftrightarrow (p_a \wedge p'_a \wedge p_e).$$

and

$$\omega_{m_j} \triangleq m_j \leftrightarrow (\neg p_a \wedge p'_a \wedge p_j).$$

We can verify that this scenario has a Nash solvable representation, stated by the validity of $\rho \wedge \omega_{m_e} \wedge \omega_{m_j} \to \Diamond_N NE$ in the logic $\Lambda(\{a, e, j\}, \{p_a, p'_a, p_e, p_j\}, \{s, m_e, m_j\})$. This is semantically clear considering that it leads to the following GPCC:

p_j.

	p_e	$\neg p_e$
$p_a \wedge p'_a$	m_e	s
$\neg p_a \wedge p'_a$	m_j	m_j
$p_a \wedge \neg p'_a$	s	s
$\neg p_a \wedge \neg p'_a$	s	s

$\neg p_j$.

	p_e	$\neg p_e$
$p_a \wedge p'_a$	m_e	s
$\neg p_a \wedge p'_a$	s	s
$p_a \wedge \neg p'_a$	s	s
$\neg p_a \wedge \neg p'_a$	s	s

It is routine to check that for every preference profile over K, there is a Nash equilibrium.

[2] Since there are two controlled atoms in the logic, a game will have four valuations. If we take a smaller set of consequences, we restrict the problem; a bigger set would have no bite.

5. CONCLUSIONS AND PERSPECTIVES

Our primary aim in this paper is to set out a tentative proposal for a formal, logical language for talking about social choice procedures.

We have seen that propositional control is a very natural way of characterising the rights and powers of players. We have seen that the operators of contingent ability form a vocabulary for expressing players' strategic deviations. This is greatly desirable for characterising many solution concepts. The basic operator of preferences is standard in modal logic, but embedded in the context of games of propositional control with consequences, it appears to be very expressive. They allow us to represent some problems of GT and SCT, namely equilibria finding and game form solvability, in terms of model checking and theorem proving respectively.

In order to pursue the agenda of developing a language for social choice procedures, we certainly need more vocabulary. There are problems of SCT that our logic is not at first sight tailored for. The models that it determines are merely strategic games, that is, a structure composed of a strategic game form with consequence in K and a preference profile over K. Most of the important properties of SCT that we would like to talk about involve a quantification over a space of game forms and preference profiles. Remarkably we applied the logic to the problem of game form solvability whose definition (Definition 18) involves such quantifications. What we did is to encode the problem of quantifying over preferences in the meta-logical problem of theorem deduction. On the side of problems with game form quantifications, in *implementation theory* [16], problems are about whether there exists a game form satisfying some desirable properties.

So, our logic is able to capture some of these features. But, it would be interesting to study other problems of social choice and pinpoint what can and cannot be formalised. A natural path of research is to investigate how to extend the framework in order to push the expressivity further without altering the essential qualities of the logic too much.

Acknowledgment. This research is funded by the EPSRC grant EP/E061397/1 *Logic for Automated Mechanism Design and Analysis (LAMDA)*. We are grateful to the reviewers of AAMAS'09.

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