Model checking strategic equilibria

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Abstract. Solution concepts are a fundamental tool for the analysis of game-like systems, and as a consequence, much effort has been devoted to the problem of characterising solution concepts using logic. However, one problem is that, to characterise solution concepts such as Nash equilibrium, it seems necessary to refer to strategies in the object language, which tends to complicate the object language. We propose a logic in which we can formulate important properties of games (and in particular pure-strategy solution concepts) without recourse to naming strategies in the object language. The idea is that instead of using predicates which state that a particular collection of strategies forms a solution, we define formulae of the logic that are true at a state if and only if this state constitutes a particular equilibrium outcome. We demonstrate the usefulness of the logic by model checking equilibria of strategic games.

1 Introduction

Game theory [18] has come to be seen as a topic of major importance for computer science, since it focuses on the study of interaction and protocols from an incentive-based perspective. *Social software* [20] aims to give social procedures a theory analogous to the formal theories for computer algorithms to facilitate, e.g., program correctness or analysis of programs. One aspect of *game logics* [23] is to study those theories with logical tools. We can distinguish two complementary families of formalisms: logics of change addressing action and time, and logics of mental states modelling informational and motivational attitudes.

Games of interaction and their solutions. A game is a description of the protocol of interaction between players and their preferences. A solution concept describes what may be the solutions (or outcomes) that emerge given some assumptions of rationality of the players.

To describe the different models of interaction, the solutions and their properties, game theory makes use of the language of mathematics which is merely set theory and plain English. One objective of game logics is to build purely logical formal languages that are able to talk about social procedures and games in particular. Some obvious merits would be to obtain unambiguous formalisations for the domain of social procedures, and the opportunity to apply formal methods of computer science to game-like systems.

Game theory is concerned with identifying sensible solutions for a particular class of game. Our present task is to propose a framework in which we can reason about them.

Model checking game solutions. The interest of the computer science community in the *agent paradigm* for software architectures is dramatically increasing, and game theory is one of the most successfully applied theories of agent interaction in computer science. As a consequence, it is not hard to argue in favour of formal methods for verifying social procedures as they are fundamental for the validation of such complex systems.

Model checking is one of these methods for hardware or software verification. A problem of model checking can be formally stated as follows: given a property (or logical formula) φ, a model $\mathcal{M}$, return the set of states S such that $s \in S$ iff φ is true at the state s in M.

One important aspect that one should have in mind when designing methods for model checking is then to provide a language of specification that will facilitate the work of the user. In this paper, we attach a particular importance to the simplicity of the syntax of our logic for the very purpose of characterising properties of games.

Action abstraction. Typically in game logics, the characterisation of solution concepts is achieved by defining predicates of the form $SC((s_i)_N)$, stating that the particular strategy profile $(s_i)_N$ is an instance of the solution concept SC[1] (for example take SC as Nash equilibrium). In such predicate definitions, strategies or actions are parameters, and so we must have a way of referring to these in the logic's object language. Propositional Dynamic Logic [14] is a natural candidate. However when this principle is integrated with logics of ability and agency like Alternating-time Temporal Logic [1], Coalition Logic [22] or STIT theories [5], there is a paradigmatic issue. Indeed, the agenda of reasoning about solution concepts seems to make it necessary to reify strategies in the object language — yet one of the putative advantages of temporal-based logics such as ATL is to abstract away from strategies and actions.

But let us take a step back, and ask the question: are explicit names of actions necessary for the logical characterisation of solution concepts? In this paper, we shall provide evidence for a negative answer. For the time being a motivational question is: what would we gain by abstracting actions away?

For model checking solution concepts we would like to give as input (1) a game, and (2) a *general* formulation of a solution concept, and obtain as output the set of outcomes that verify it. For the existing logics able to express game equilibria, the straightforward way of applying model checking to verifying solution concepts is somewhat limited because the modeller first has to choose an action profile $(s_i)_N$ and then check whether the game satisfies $SC((s_i)_N)$. Either the hard work is done by the designer, in selecting the action profile, or we need to provide to the model checker a large formula containing as many disjunctions as the model to be tested has strategy profiles. This leads to a formula exponentially large in the number of strategies. As we will see, our definitions of solution concept are not subject to this drawback. We thus can characterise important properties of game in a more succinct manner. This is also desirable since the complexity of model checking typically depends on the size of the input formula.

Of course, abstraction of action names is not a solution to every problem in social software. For a completely different perspective, see [24] in which the author considers

[1] We call N the *grand coalition*, the coalition containing all players. A *strategy profile* is a combination strategies: one for each player in N.

strategies to be “the unsung heroes of game theory”. However, we show in what follows that without relying on explicit actions, we are able to give a general logical formulation *SC* for most solution concepts in strategic games. As a consequence, we can check in a very natural manner where the equilibria are in a game.

Outline. This article aims at providing a language for characterising properties of games, which is expressive, easy to manipulate, unambiguous, and in this sense particularly suitable for a designer of interaction protocols in need of a tool for model checking their game theoretic properties. We first introduce some concepts from game theory and some solution concepts. Next, we present our logic and characterise the solution concepts in it. We continue with examples. We conclude with an informal discussion and perspectives.

2 Some notions from Game Theory

In this section, we review the basics of game theory in strategic games.

2.1 Strategic games

Definition 1 (strategic game form). *A* strategic game form *is a tuple* $\langle N, (A_i) \rangle$ *where:*

- N *is a finite set of* players *(or agents);*
- A_i *is a nonempty set of* actions *for each player* $i \in N$.

A strategic game form is sometimes called a *mechanism*. It specifies the agents taking part in the game and the actions available to them. Next, we need preferences, which will give the players the incentive for taking an action.

Definition 2 (preference relation). *A* preference relation $\succeq$ *over S is a total, transitive and reflexive binary relation over S.*

We can now see a strategic game as basically the composition of a strategic game form with a collection of preference relations (one for every agent).

Definition 3 (strategic game). *A* strategic game *is a tuple* $\langle N, (A_i), (\succeq_i) \rangle$ *where* $\langle N, (A_i) \rangle$ *is a strategic game form, and for each player* $i \in N$, $\succeq_i$ *is a* preference relation *over* $A = \times_{j \in N} A_j$.

We refer to a collection $(a_j)_{j \in N}$, consisting of one action for every agent in N, as an *action profile*. Given an action profile a, we denote by a_i the action of the player i, and by a_{-i} the action of the coalition $N \setminus \{i\}$. We write a_C for the *coalitional actions* that are members of $A_C = \times_{j \in C} A_j$ for any $C \subseteq N$.

Strategic games are models of interaction in which all players choose an action simultaneously and independently. It is convenient to see the elements of A as the outcomes of the game, resulting from an action profile. There are three ingredients that are characteristic of *game theoretic* interactions in strategic games: (i) agents are independent, in the sense that every player i can freely decide which move in A_i to take whatever the other agents choose – all combinations of agents' choices are compatible; (ii) not only those combinations are compatible, but they also lead to a unique outcome (here formally represented by the action profile itself); and (iii) the preferences $\succeq_i$ are over the possible outcomes A, which gives the game theoretic flavour: players must take into account the preferences of others in order to determine how to achieve the best outcome for themselves.

	a_2	b_2
a_1	$1, 1$	$2, 0$
b_1	$0, 2$	$0, 0$

Fig. 1. An example of 2-player strategic game.

In 2-player games, it is convenient to represent a strategic game as a matrix of utilities (or payoffs). In the game shown in Figure 1, player 1 is the row player, choosing between action a_1 and b_1, and player 2 is the column player, choosing between action a_2 and b_2. The entries (x, y) of the matrix represent the payoffs of agents for a particular outcome — x is the payoff for the row player, while y is the payoff for the column player. The preferences are easily derived. For example $(a_1, a_2) \succeq_1 (b_1, a_2)$, $(a_1, a_2) \succeq_2 (a_1, b_2)$ but $(a_1, a_2) \not\succeq_1 (a_1, b_2)$, and $(b_1, b_2) \succeq_2 (a_1, b_2)$ and $(a_1, b_2) \succeq_2 (b_1, b_2)$.

2.2 Game equilibria

Next, we define some important solution concepts in *pure strategies*. Those are definitions of very standard notions of game theory. We refer the reader to [18]. We will later demonstrate the ability of our logic to represent properties of strategic games, and game equilibria in particular. In order to show how fine-grained the logic is, we will study several variants of equilibria, namely two sorts of Pareto optimality, three sorts of dominance, two sorts of Nash equilibria and the concept of the core.

Definition 4 (Pareto optimality). *An action profile a^* is a* weak Pareto optimum *if there is no action profile strictly preferred over a^* by every agent. a^* is a* strong Pareto optimum *if there is no action profile considered at least as good as a^* by every agent and strictly preferred by at least one agent.*

Definition 5 (dominance equilibria). *a^* is a* very weakly dominant *action profile if for every player i and coalitional action a_{-i}, i considers (a_i^*, a_{-i}) at least as good as (a_i', a_{-i}) for every a_i'. a^* is a* weakly dominant *action profile if for every agent i, one preference is strict for at least one action a_i'. a^* is a* strictly dominant *action profile if all preferences are strict.*

Definition 6 (Nash equilibrium). *An action profile $a^* \in A$ is a* Nash equilibrium *iff for every player $i \in N$ and for all $a_i \in A_i$, i considers (a_{-i}^*, a_i^*) at least as good as (a_{-i}^*, a_i).*

To conclude this collection of solution concepts, we will also show interest in cooperative games via the study of strong Nash equilibrium and the core of strategic games.

Definition 7 (strong Nash equilibrium). *An action profile a^* is a* strong Nash equilibrium *of a strategic game iff there is no coalition $C \subset N$ and no strategy a_C such that (a_C, a^*_{-C}) is considered strictly better than a^* by every player of C.*

Definition 8 (weak core membership). *An action profile a^* is* dominated *in a strategic game iff there is a coalition $C \subset N$ and a strategy a_C such that for all a_{-C}, every $i \in C$ strictly prefers (a_C, a_{-C}) over a^*. a^* is in the* weak core *of the game if it is not dominated.*

These last two definitions hold for a *coalitional game without transferable utilities*. That is, players can form coalitions, but cannot redistribute the sum of the payoffs among the individuals of the coalition.

3 A Hybrid Logic of Choice and Preference

We now introduce a logic that will allow us to capture game theoretic solution concepts such as those above, without recourse to naming strategies/actions in the object language.

At the heart of the models we use *Kripke frames*: we assume a set of states and binary relations over them. We will think of a state as an action profile. For any coalition J, an equivalence relation R_J will cluster together the states that J cannot separate by one of its choices: the exact outcome will depend on the choice of the other agents which is out of control of J. The main task is to constrain the frames $\langle S, (R_J) \rangle$ such that they are a correct conceptualisation of strategic game forms. We will also have a preference relation P_i for every agent i. This logic is a hybrid logic [3], and in what follows, we pre-suppose some familiarity with this class of formalisms.

3.1 Language and semantics

Let us assume $\mathcal{A}gt = \{0, 1 \ldots n\}$ a nonempty finite set of *agents*, $\mathcal{P}rop = \{p_1, p_2 \ldots\}$ a countable set of *propositions*, $\mathcal{N}om = \{i_1, i_2 \ldots\}$ a countable set of *nominals* and $\mathcal{W}Var = \{x_1, x_2 \ldots\}$ a countable set of *state variables*. $\mathcal{P}rop, \mathcal{N}om, \mathcal{W}Var$ are pairwise disjoint. We call $\mathcal{S}ymb = \mathcal{N}om \cup \mathcal{W}Var$ the set of *state symbols*. The set of atoms is then denoted $\mathcal{A}tm = \mathcal{P}rop \cup \mathcal{S}ymb$.

The syntax of HLCP is defined by the BNF

$$\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid [J]\varphi \mid [\preceq_i]\varphi \mid @_s\varphi \mid\downarrow x.\varphi$$

where $a \in \mathcal{A}tm$, $x \in \mathcal{W}Var$, $s \in \mathcal{S}ymb$, $i \in \mathcal{A}gt$ and $J \subseteq \mathcal{A}gt$ are terminal symbols. This is a multi-modal language of hybrid logic with $@$ and $\downarrow$ (from now on $\mathcal{H}(@, \downarrow)$).

As usual, the remaining Boolean connectives are defined by abbreviations, and $\langle J \rangle\varphi =_{def} \neg[J]\neg\varphi$. Analogously, $\langle \preceq_i \rangle\varphi =_{def} \neg[\preceq_i]\neg\varphi$. In the object language, we denote by $\overline{J}$ the complement of J w.r.t. $\mathcal{A}gt$.

The intended reading of $[J]\varphi$ is "group J chooses such that φ whatever other agents do" or "the current choices of agents in J ensure that φ". $\langle J\rangle\varphi$ is "J by its current choice does not rule out φ as a possible outcome" or "J allows φ". In particular, because the empty coalition cannot make any choice (or more precisely has a unique vacuous choice), $[\emptyset]\varphi$ can be read as "φ cannot be avoided" and $\langle\emptyset\rangle\varphi$ reads "φ is a possible outcome". $\langle\preceq_i\rangle\varphi$ means that at the current state, i prefers φ or is indifferent about it. $@_s\varphi$ means that φ is true at the state labelled s. Finally, the operator $\downarrow x.$ labels the current state with the state variable x. Then it allows further explicit reference to the state by using x as an atom in the formula in its scope.

Definition 9 (HLCP model). *A model for HLCP is a tuple* $\langle \mathcal{A}gt, \mathcal{P}rop, \mathcal{N}om, \mathcal{W}Var, S, (R_J), (P_i), \pi\rangle$ *where:*

- *$\mathcal{A}gt$, $\mathcal{P}rop$, $\mathcal{N}om$ and $\mathcal{W}Var$ are as before;*
- *S is a set of states;*
- *every R_J is an* equivalence relation *over S such that:*
  - *(1.)* $R_{J_1\cup J_2} \subseteq R_{J_1}$;
  - *(2.)* $R_{J_1}\cap R_{J_2} \subseteq R_{J_1\cup J_2}$;
  - *(3.)* $R_\emptyset \subseteq R_J \circ R_{\mathcal{A}gt\setminus J}$;
  - *(4.)* $R_{\mathcal{A}gt} = Id$;
- *every P_i is a total, transitive relation over S;*
- $\pi : S \longrightarrow 2^{\mathcal{P}rop\cup\mathcal{N}om}$ *is a valuation function where* $\pi^{-1}(i)$ *is a singleton for every* $i \in \mathcal{N}om$.

An assignment, *g, is a mapping from $\mathcal{S}ymb$ into S. We define g^x_s as $g^x_s(x) = s$ and $g^x_s(x) = g(y)$ for $x \neq y$.*

The definition of valuation function of our models is conceptually important here.

The fact that the valuation of a nominal is a singleton reflects the main aspect of hybrid logic. A nominal uniquely characterises a state in the Kripke model and can thus be understood as the name of a state.

An equivalence relation in a Kripke model generates a partition of the set of states. Hence, the relation R_J represents the choices of J, and each element of the underlying *partition* (viz. a set of states) corresponds to one choice. We will state this formally in Section 4.1. $R_\emptyset$ represents the choice of the empty coalition. Since the empty coalition is assumed to have only one ubiquitous choice, $R_\emptyset$ is the universal relation over the possible outcomes. (1.) means that adding agents to a coalition makes it at least as effective. (2.) means that a coalition is not more effective than the combination of its parts. (3.) says that an outcome is possible only if one can reach it by two successive moves along two relations of choice of two complementary coalitions. This is intended to reflect the independence of agents. (4.) means that the grand coalition is maximally effective: if an outcome is possible then the grand coalition can choose it deterministically.

An example of a *malformed* HLCP (pre-)model with two players is given in Figure 2. Here, the relation $R_{\{0\}}$ partitions the set of outcomes into two partitions, each consisting of four states. The relation $R_{\{1\}}$ partitions the set of outcomes in three partitions. The relation $R_{\mathcal{A}gt} = R_{\{0,1\}}$ is represented by dashed lines. The relation $R_\emptyset = R_{\{0\}}\circ R_{\{1\}}$ groups the outcomes of a strategic game together. At s, the constraint (2.) on R is not

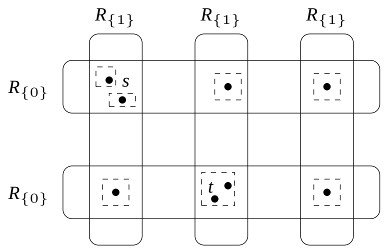

Fig. 2. Dashed lines represent $R_{\mathcal{A}gt}$ relations. This is *not* an HLCP model.

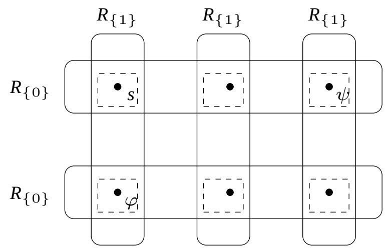

Fig. 3. An example of HLCP pre-model. Preferences are not represented.

satisfied. At t, the constraint (4.) on R is not satisfied. An example of HLCP (pre-)model with two players is represented in Figure 3.

Truth values are given by:

– $\mathcal{M}, g, s \models p$ iff $p \in \pi(s)$, for $p \in \mathcal{P}rop$
– $\mathcal{M}, g, s \models t$ iff $g(t) = s$, for $t \in \mathcal{S}ymb$
– $\mathcal{M}, g, s \models @_t\varphi$ iff $\mathcal{M}, g, g(t) \models \varphi$, where $t \in \mathcal{S}ymb$
– $\mathcal{M}, g, s \models \downarrow x.\varphi$ iff $\mathcal{M}, g^x_s, s \models \varphi$
– $\mathcal{M}, g, s \models [J]\varphi$ iff for all $s' \in R_J(s),\ \mathcal{M}, g, s' \models \varphi$
– $\mathcal{M}, g, s \models [\preceq_i]\varphi$ iff for all $s' \in P_i(s),\ \mathcal{M}, g, s' \models \varphi$

and as usual for classical connectives. We also adopt the conventional definitions of satisfiability and validity: an HLCP formula φ is *satisfiable* iff there exists a pointed model $\mathcal{M}, g, s$ such that $\mathcal{M}, g, s \models \varphi$ and φ is *valid* iff for every pointed model $\mathcal{M}, g, s$ we have $\mathcal{M}, g, s \models \varphi$.

We shall write $\mathcal{M}, s \models \varphi$ when it is the case that $\mathcal{M}, g, s \models \varphi$ for any mapping g.

3.2 Some intuitions about the logic

The frames of HLCP models are the frames we expect for studying strategic games. Definition 9 item (3.) defines the powers of the empty coalition and reflects the independence of agents. A state is considered possible if it can be reached via the relation $R_\emptyset$. A state is possible only if it is compatible with the choices of complementary coalitions. From items (2.) and (4.), the pointwise intersections of agents' classes of choice are singletons: $\bigcap_{i \in \mathcal{A}gt} R_{\{i\}}(s) = \{s\}$ for every s. Hence, possible states map directly to action profiles. We explain this in more detail now.

Actions and choices explained Given a coalition J, and two states s and s' in S, $s \in R_J(s')$ means that s and s' are two possible outcomes of a same choice of J. By definition (Definition 9 item (2.)), a choice of a coalition is the intersection of the choices of its individual members. Hence $s \in R_J(s')$ means that s and s' are in a same choice of every agent in J. To put it another way, no agent in J can choose (resp. dismiss) s without choosing (resp. dismissing) s'.

The operator $\langle\overline{J}\rangle$ allows to quantify over possible states, given that the actions of the agents out of J are fixed. Equivalently, keeping in mind the analogy of states as action profiles, it makes it possible to quantify over actions of J. For example, $\langle\overline{\{i\}}\rangle$ quantifies over i's actions. $\langle\overline{J}\rangle\varphi$ can be read "the action of the agents that are not in J being maintained, there is an action of J such that φ".

In the model of Figure 3 with $\mathcal{A}gt = \{0, 1\}$, at state s, player 1 can unilaterally change its current choice such that ψ holds: $\mathcal{M}, s \models \langle\overline{\{1\}}\rangle\psi$, or equivalently $\mathcal{M}, s \models \langle\{0\}\rangle\psi$, meaning that player 0 allows ψ. Analogously player 1 allows φ: $\mathcal{M}, s \models \langle\{1\}\rangle\varphi$. Hence player 0 can change its choice such that φ holds: $\mathcal{M}, s \models \langle\overline{\{0\}}\rangle\varphi$.

The action component of the logic is largely inspired by the Chellas's STIT logic [15]. The logic limited to individuals has been axiomatised by Xu [5, Chap. 17] and studied further in [4]. [9] proposes a group version of the logic. However, the models are more general than those of the original logic of [15]. They allowed for example what is exemplified in Figure 2 at s, that is to say an imperfect intersection of the relations of choice. As a consequence a coalition is more effective than the set of its parts. However, this is not compatible with strategic games. Hence, in HLCP we forced this by the constraint (2.) that is not present in the models of [9].

Then, like STIT, HLCP has obvious links with multi-agent epistemic logic [11] and multi-dimensional logics over equivalence relations [17]. It may indeed be helpful to think about a choice relation as an epistemic relation. In epistemic logics, $[i]\varphi$ would read "i knows that φ". For a coalition J, $[J]$ is similar to the *distributed knowledge* operator of epistemic logic, usually written D_J. Alternatively, we could have used cylindric modal logic [27] or logics of propositional control [26, 13]. In these logics, a formula of the form $\Diamond_J\varphi$ reads the agents in J can change their choice such that φ holds. It trivially corresponds in HLCP to the formula $\langle\overline{J}\rangle\varphi$ meaning that the agents out of J allow for φ.

About the hybridisation Intuitively, $\downarrow x.\varphi$ assigns the name of the current state to the variable x, and it can be reused in the scope of the binder as a propositional letter. The authors of [2] compare the role of the binder $\downarrow$ to the Reichenbachian *generalised present tense*. They write:

> It enables us to "store" an evaluation point, thereby making it possible to insist later that certain events happened at *that* time, or that certain other events must be viewed from that particular perspective. This is precisely the kind of expressive power we need to encode Reichenbach's ideas.

We argue that this ability to fix an 'evaluation point', viz. an action profile in our setting, and looking at alternatives from that perspective, is also precisely what we need to encode most game equilibria.

We can already take advantage of the power of hybrid logic for defining strict preferences which will be useful later.

Definition 10 (strict preferences). *The* strict preference *of i for an alternative where φ holds is defined by:*

$$\langle\prec_i\rangle\varphi =_{def} \downarrow x.\langle\preceq_i\rangle(\varphi \wedge \neg\langle\preceq_i\rangle x)$$

Note that the expressive power of hybrid logic makes it possible to characterise in the object language some features of models in a way that is not possible in conventional modal logic. For instance, the ability to grasp the intersection of relations was a key trigger for the modern era of hybrid logic [21]. This leads us to the axiomatic characterisation of HLCP.

3.3 Axiomatisation

x will be used as a meta-variable over the set of state variables $\mathcal{W}Var$; s, t and u will be meta-variables over the set of state symbols $\mathcal{S}ymb$; and $\boxdot$ is any modality from $\{[J] \mid J \subseteq \mathcal{A}gt\} \cup \{[\preceq_i] \mid i \in \mathcal{A}gt\}$.

There exist several presentations of the axiomatics of the basic hybrid logic with $@$ and $\downarrow$ (hereafter $\mathbb{K}_{\mathcal{H}(@,\downarrow)}$) [7, 6, 3]. We use one given in [6] which unlike the others we can find in the literature, does not have recourse to unorthodox rules, viz. rules of inference that apply under syntactic constraints. We show it in Figure 4. Note that a substitution replaces uniformly (1) proposition variable by arbitrary formulae and (2) nominals by other nominals.

axioms:	
(CT)	enough classical tautologies
($K_\boxdot$)	$\boxdot(p \to q) \to (\boxdot p \to \boxdot q)$
($K_@$)	$@_s(p \to q) \to (@_s p \to @_s q)$
($\text{selfdual}_@$)	$@_s p \leftrightarrow \neg @_s \neg p$
($\text{ref}_@$)	$@_s s$
(agree)	$@_t @_s p \leftrightarrow @_s p$
(intro)	$s \to (p \leftrightarrow @_s p)$
(back)	$\neg \boxdot \neg @_s \varphi \to @_s \varphi$
(DA)	$@_s(\downarrow x.\varphi \leftrightarrow \varphi[x/s])$
($\text{name}_\downarrow$)	$\downarrow x.(x \to \varphi) \to \varphi$, provided that x does not occur in φ
($BG_\downarrow$)	$@_s \boxdot \downarrow x.@_s \neg \boxdot \neg x$
rules:	
(MP)	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
(subst)	From $\vdash \varphi$ infer $\vdash \varphi^\sigma$, for σ a substitution
($\text{nec}_@$)	From $\vdash \varphi$ infer $\vdash @_s \varphi$
($\text{nec}_\downarrow$)	From $\vdash \varphi$ infer $\vdash \downarrow x.\varphi$
($\text{nec}_\boxdot$)	From $\vdash \varphi$ infer $\vdash \boxdot \varphi$

Fig. 4. An axiomatisation of $\mathbb{K}_{\mathcal{H}(@,\downarrow)}$.

The principles are sound and axiomatise completely $\mathbb{K}_{\mathcal{H}(@,\downarrow)}$ when the operators symbolised by $\boxdot$ are normal modalities over arbitrary frames (i.e., K-modalities). We now need to give the principles that will ensure that the modalities of the form $[\preceq_i]$ represent a relation of preference and the collection of modalities of the form $[J]$ represent a strategic game form.

We say a formula is *pure* if it contains no propositional variables (but may contain nominals). We obtain the full axiomatisation of HLCP by adding the pure axiom schemata listed in Figure 5. It is easy to check that these principles are sound. An important theorem of hybrid logic states that if Σ is a set of pure $\mathcal{H}(@,\downarrow)$ formulae, then $\mathbb{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$ is complete for the class of frames on which each formula of Σ is valid [7, Th. 4.11]. Proving the completeness of the inference system is thus straightforward.

($T_{[J]}$)	$s \rightarrow \langle J \rangle s$
($5_{[J]}$)	$\langle J \rangle s \rightarrow [J]\langle J \rangle s$
(mon)	$\langle J_1 \cup J_2 \rangle s \rightarrow \langle J_1 \rangle s$
(inter)	$\langle J_1 \rangle s \wedge \langle J_2 \rangle s \rightarrow \langle J_1 \cup J_2 \rangle s$
($\text{elim}_{[\emptyset]}$)	$\langle \emptyset \rangle s \rightarrow \langle J \rangle \langle \overline{J} \rangle s$
($\text{det}_{[Agt]}$)	$\langle \mathcal{A}gt \rangle s \rightarrow s$
($4_{[\preceq_i]}$)	$\langle \preceq_i \rangle \langle \preceq_i \rangle s \rightarrow \langle \preceq_i \rangle s$
(total)	$s \wedge \langle \emptyset \rangle t \rightarrow \langle \preceq_i \rangle t \vee @_t \langle \preceq_i \rangle s$

Fig. 5. Principles added to the axiomatisation of $\mathbb{K}_{\mathcal{H}(@,\downarrow)}$, completing the axiomatisation of HLCP.

We try to give intuitive readings of the axioms of Figure 5. ($T_{[J]}$) means that if s is the state at hand, it is in the current choice of everyone. ($5_{[J]}$) means that for every coalition J, if J allows s then J refuse not to allow it. (mon) expresses the fact that if a group allows s then its parts allow s also. (inter) means that if some parts allow s then the coalition composed of these parts allows s too. (elim) means that if an outcome is possible then a coalition always allows that its complementary coalition could allow s too. (det) captures the fact that if the grand coalition allows s then s is the outcome. ($4_{[\preceq_i]}$) and (total) are intuitively the transitivity and connectedness of preferences.

Proposition 1 (completeness). *HLCP is complete with respect to the class of HLCP models.*

PROOF. By applying the Standard Translation (ST) for hybrid logic, we can check that the pure axioms in the last tabular correspond to the constraints we imposed on the frames. The correspondence is pretty clear for whom is familiar with the ST for hybrid logic. (Or modal logic: just recall that a state symbol is true exactly at one state.) As an example, we nevertheless give the translation for (inter) and (total). (The subscript t is a state symbol that does not occur in the formula being translated.)

– (inter) corresponds to the constraint $R_{J_1} \cap R_{J_2} \subseteq R_{J_1 \cup J_2}$:
  - $ST_t(\langle J_1 \rangle s \wedge \langle J_2 \rangle s \rightarrow \langle J_1 \cup J_2 \rangle s)$;
  - $ST_t(\langle J_1 \rangle s) \wedge ST_t(\langle J_2 \rangle s) \rightarrow ST_t(\langle J_1 \cup J_2 \rangle s)$;
  - $\exists y_1.(R_{J_1}(t,y_1) \wedge ST_{y_1}(s)) \wedge \exists y_1.(R_{J_2}(t,y_2) \wedge ST_{y_2}(s)) \rightarrow$
    $\exists y_1.(R_{J_1 \cup J_2}(t,y_3) \wedge ST_{y_3}(s))$;
  - $\exists y_1.(R_{J_1}(t,y_1) \wedge (y_1 = s)) \wedge \exists y_1.(R_{J_2}(t,y_2) \wedge (y_2 = s)) \rightarrow$
    $\exists y_1.(R_{J_1 \cup J_2}(t,y_3) \wedge (y_3 = s))$;

- $R_{J_1}(t,s) \wedge R_{J_2}(t,s) \rightarrow R_{J_1 \cup J_2}(t,s)$.

– (total) corresponds to the constraint "P_i is total":
  - $ST_u(s \wedge \langle\emptyset\rangle t \rightarrow \langle\preceq_i\rangle t \vee @_t\langle\preceq_i\rangle s)$;
  - $ST_u(s) \wedge ST_u(\langle\emptyset\rangle t) \rightarrow ST_u(\langle\preceq_i\rangle t) \vee ST_u(@_t\langle\preceq_i\rangle s)$;
  - $(u = s) \wedge \exists y_1.(R_\emptyset(u,y_1) \wedge ST_{y_1}(t)) \rightarrow$
    $\exists y_2.(P_i(u,y_2) \wedge ST_{y_2}(t)) \vee \exists y_3.(P_i(t,y_3) \wedge ST_{y_3}(s))$;
  - $(u = s) \wedge \exists y_1.(R_\emptyset(u,y_1) \wedge (y_1 = t)) \rightarrow$
    $\exists y_2.(P_i(u,y_2) \wedge (y_2 = t)) \vee \exists y_3.(P_i(t,y_3) \wedge (y_3 = s))$;
  - $R_\emptyset(s,t) \rightarrow P_i(s,t) \vee P_i(t,s)$.

HLCP only consists of a set of pure axiom schemata added to the axiomatisation of $\mathbb{K}_{\mathcal{H}(@,\downarrow)}$. Hence, the result follows as a corollary of [7, Th. 4.11]. ■

4 Application to game analysis

In the introduction to the paper, we promised that we would formalise solution concepts without using names for actions. In this section, we make good on that promise. We show how to characterise a number of solution concepts using the logic.

4.1 Relating strategic games and HLCP models

We here guarantee that HLCP models are an adequate conceptualisation of strategic games. With this aim, we relate strategic games $G = \langle N, (A_i), (\succeq_i)\rangle$ with the models of HLCP. Let us first introduce a hybrid version of strategic games.

Definition 11 (hybrid game model). *A* hybrid game model *is a tuple* $\langle N, (A_i), (\succeq_i), \mathcal{P}rop, \mathcal{N}om, \mathcal{WV}ar, v\rangle$ *where* $\langle N, (A_i), (\succeq_i)\rangle$ *is a strategic game,* $\mathcal{P}rop, \mathcal{N}om$ *and* $\mathcal{WV}ar$ *are as in Definition 9, and v maps elements from* $\times_{i\in N} A_i$ *to* $2^{\mathcal{P}rop \cup \mathcal{N}om}$.

Hybrid game models are strategic games with propositions and a function of interpretation, to which we add the standard 'hybrid machinery'. They are sufficiently rich to give a semantics to the language of HLCP. Truth values of HLCP formulae over hybrid game models are defined recursively as follows.

Definition 12 (truth values in hybrid game models). *Let a hybrid game model* $\mathcal{M}_G = \langle N, (A_i), (\succeq_i), \mathcal{P}rop, \mathcal{N}om, \mathcal{WV}ar, v\rangle$. *Let g, be a mapping from* $\mathcal{S}ymb$ *into A as in Definition 9.*

– $\mathcal{M}_G, g, a \models_{sg} p$ *iff* $p \in v(a)$, *for* $p \in \mathcal{P}rop$
– $\mathcal{M}_G, g, a \models_{sg} t$ *iff* $g(t) = a$, *for* $t \in \mathcal{S}ymb$
– $\mathcal{M}_G, g, a \models_{sg} @_t\varphi$ *iff* $\mathcal{M}_G, g, g(t) \models_{sg} \varphi$, *where* $t \in \mathcal{S}ymb$
– $\mathcal{M}_G, g, a \models_{sg} \downarrow x.\varphi$ *iff* $\mathcal{M}_G, g^x_a, a \models_{sg} \varphi$
– $\mathcal{M}_G, g, a \models_{sg} [J]\varphi$ *iff for every* $a'_{-J} \in \times_{j\in N\setminus J} A_j$ *we have* $\mathcal{M}_G, g, (a_J, a'_{-J}) \models_{sg} \varphi$
– $\mathcal{M}_G, g, a \models_{sg} [\preceq_i]\varphi$ *iff for every* $a' \succeq_i a$ *we have* $\mathcal{M}_G, g, a' \models_{sg} \varphi$

and as usual for classical connectives.

We say a HLCP formula φ is *sg-satisfiable* iff there exists a pointed hybrid game model $\mathcal{M}_G, g, a$ such that $\mathcal{M}_G, g, a \models_{sg} \varphi$ and φ is *sg-valid* iff for every pointed hybrid game model $\mathcal{M}_G, g, a$ we have $\mathcal{M}_G, g, a \models_{sg} \varphi$.

From a hybrid game model we obtain a corresponding HLCP model as follows.

Definition 13 (from hybrid game models to HLCP models). *We say an HLCP model* $\langle \mathcal{A}gt, \mathcal{P}rop, \mathcal{N}om, \mathcal{WV}ar, S, (R_J), (P_i), \pi \rangle$ corresponds *to a hybrid game model* $\langle N, (A_i), (\succeq_i), \mathcal{P}rop, \mathcal{N}om, \mathcal{WV}ar, v \rangle$ *if:*

- $\mathcal{A}gt = N$;
- $S = \times_{i \in N} A_i$;
- $(a_J, a_{-J}) R_J (a_J, a'_{-J})$;
- $a' \in P_i(a)$ *iff* $a' \succeq_i a$;
- $\pi = v$.

It was already clear that we conceive a state in HLCP as an action profile. Two action profiles are in the same class of choice of J if agents in J do the same action in both profiles; preferences are immediate.

The other way round, we could construct a hybrid game model corresponding to an HLCP model. We just give it for clarification but will not make use of it.

Definition 14 (from HLCP models to hybrid games models). *We say that a hybrid game model* $\langle N, (A_i), (\succeq_i), \mathcal{P}rop, \mathcal{N}om, \mathcal{WV}ar, v \rangle$ corresponds *to an HLCP model* $\langle \mathcal{A}gt, \mathcal{P}rop, \mathcal{N}om, \mathcal{WV}ar, S, (R_J), (P_i), \pi \rangle$ *if:*

- $N = \mathcal{A}gt$;
- $A_i = S|_{\equiv R_{\{i\}}} = \{|s|_{\equiv R_{\{i\}}} : s \in S\}$;
- $(|s_0|_{\equiv R_{\{0\}}}, \ldots |s_k|_{\equiv R_{\{k\}}}) \succeq_i (|s'_0|_{\equiv R_{\{0\}}}, \ldots |s'_k|_{\equiv R_{\{k\}}})$ *iff* $y \in P_i(x)$, *where* $k = Card(\mathcal{A}gt) - 1$, $x \in \bigcap_{i \in \mathcal{A}gt} |s'_i|_{\equiv R_{\{i\}}}$ *and* $y \in \bigcap_{i \in \mathcal{A}gt} |s_i|_{\equiv R_{\{i\}}}$;
- $v = \pi$.

The notation makes it perhaps less self explanatory than the previous definition. It identifies the set of actions of an agent i with the set of classes in the equivalence relation of choice $R_{\{i\}}$. An action profile is then captured by a tuple of such classes of choice, one for every agent. As a consequence of the items 2 and 4 of Definition 9, the classes of choice in a tuple intersect in exactly one state: thus, x and y in the definition above are uniquely determined. The preferences in the strategic game model are then derived from the relation P_i applied to this state.

4.2 Equilibria in HLCP models

Our next task is to adapt the previous definitions of equilibria in the context of HLCP models. We also state their correspondence with the game theoretic definitions.

Definition 15. *Given an HLCP model* $\mathcal{M}$ *and a state* s^* *in* $\mathcal{M}$, s^* *is* weakly Pareto optimal *iff there is no* $s \in R_\emptyset(s^*)$ *such that* $s \in P_i(s^*)$ *and* $s^* \notin P_i(s)$ *for every* i *in* $\mathcal{A}gt$. s^* *is* strongly Pareto optimal *iff there is no* $s \in R_\emptyset(s^*)$ *such that* $s \in P_i(s^*)$ *for every* i *in* $\mathcal{A}gt$, *and there is a* j *such that* $s^* \notin P_j(s)$.

Definition 16. *Given an* HLCP *model* $\mathcal{M}$ *and a state* s^* *in* $\mathcal{M}$*,* s^* *is*

1. very weakly dominant *iff for all* i *in* $\mathcal{A}gt$ *and for all* $s \in R_i(s^*)$*, we have that for all* $s' \in R_{\mathcal{A}gt\setminus\{i\}}(s)$*,* $s \in P_i(s')$*;*
2. weakly dominant *iff for all* i *in* $\mathcal{A}gt$ *and for all* $s \in R_i(s^*)$*, we have that for all* $s' \in R_{\mathcal{A}gt\setminus\{i\}}(s)$*,* $s \in P_i(s')$ *and there is a* $s'' \in R_{\mathcal{A}gt\setminus\{i\}}(s)$ *such that* $s'' \notin P_i(s)$*;*
3. strictly dominant *iff for all* i *in* $\mathcal{A}gt$ *and for all* $s \in R_i(s^*)$*, we have that for all* $s' \in R_{\mathcal{A}gt\setminus\{i\}}(s)$*,* $s \in P_i(s')$ *and* $s' \notin P_i(s)$*.*

Definition 17. *Given an* HLCP *model* $\mathcal{M}$ *and a state* s^* *in* $\mathcal{M}$*,* s^* *is*

1. Nash equilibrium *iff for all* i *in* $\mathcal{A}gt$*, for all* s *in* $R_{\mathcal{A}gt\setminus\{i\}}(s^*)$ *we have* $s \in P_i(s^*)$*;*
2. strong Nash equilibrium *iff for all* $J \subset \mathcal{A}gt$ *and* $s \in R_{\mathcal{A}gt\setminus J}(s^*)$ *there is an* i *in* J *such that* $s^* \in P_i(s)$*.*

Definition 18. *Given an* HLCP *model* $\mathcal{M}$ *and a state* s^* *in* $\mathcal{M}$*,* s^* *is in the weak* core *iff for all* $J \subset \mathcal{A}gt$ *and* $s \in R_{\mathcal{A}gt\setminus J}(s^*)$ *there is an* i *in* J *and an* $s' \in R_J$ *such that* $s^* \in P_i(s')$*.*

These definitions are adequate with the definition of game theory. This is stated by the next proposition.

Proposition 2. *Let SC be a solution concept among weakly Pareto optimal, strongly Pareto optimal, very weakly dominant, weakly dominant, strictly dominant, Nash equilibrium, strong Nash equilibrium and core. Given a hybrid strategic game* $\mathcal{M}_G$ *and a corresponding* HLCP *model* $\mathcal{M}$*, an action profile of* $\mathcal{M}_G$ *is SC iff it is an SC in* $\mathcal{M}$*.*

We will rely next on the definitions in terms of relational models introduced in this section for implementing the solution concepts in the language of HLCP.

4.3 Implementation of equilibria in HLCP

This section provides 'constant predicates' characterising that a state is a particular solution concept. To put it another way, we give context-free definitions of solution concepts in the language of HLCP. We start by defining predicates for *best response* (weak and strict). Informally, the best response of the agent i is the strategy in the repertoire of i that is most favorable to i when the strategies of the other players are given. It will be instrumental in the definition of Nash equilibrium and dominance equilibria next. As a simple illustration, we also characterise the concept of *never best response*.

WBR_i is intended to read "i plays a weak best response to the other agents' choice in the current state" by what could be reworded as "the other agents choose that i considers the current state at least as good". Formally,

$$WBR_i =_{def} \downarrow x.[\overline{\{i\}}]\langle \preceq_i \rangle x$$

We see how "binding" the current state to the variable x permits us to use it such that $[\overline{\{i\}}]x$ exactly quantifies over the alternatives allowed by the current choice of the other agents (agents in $\mathcal{A}gt \setminus \{i\}$) at the state recorded in x. Since the grand coalition is deterministic, i itself is the 'final chooser'. i plays its best response if in every alternative allowed by the other agents' current choice, i would consider its current choice at least as good.

The notion of strict best response is obtained by replacing the weak preference modality by the strict one, and since $[\overline{\{i\}}]$ is reflexive, we need to use a conditional such that the current state (obviously not strictly preferred) is not compared.[2]

$$SBR_i =_{def} \downarrow x.[\overline{\{i\}}](\neg x \rightarrow \langle \prec_i \rangle x)$$

We can use WBR_i and SBR_i as the building blocks for defining more complex notions. Before focusing on several equilibria, we can see for example that the notion of a choice that is never a best response is intuitively captured in our language, using an agentive formula stating that i chooses that it does not play a weak best response (whatever other agents do):

$$NBR_i =_{def} [i]\neg WBR_i$$

The current choice of an agent i is never a best response if i chooses that it does not play a best response (whatever other agents do). A choice that is never a best response (or equivalently which is always dominated) are often worth considering in game theory because a rational player will never use such a choice: it would always be better off choosing the strategy that dominates it.

In the remaining of this section, we give the characterisation of every solution concept defined previously.

Pareto optimality A state is a *weak Pareto optimum* if there is no other state that makes every agent better off.

$$WPO =_{def} \downarrow x.[\emptyset] \bigvee_{i \in \mathcal{A}gt} \langle \preceq_i \rangle x$$

A state labelled x is a strong Pareto optimum if there is no state y that is considered by every agent at least as good as x and which is strictly preferred by at least one agent. We can formulate this as:

$$SPO =_{def} \downarrow x.[\emptyset](\downarrow y.(@_x \bigwedge_{i \in \mathcal{A}gt} \langle \preceq_i \rangle y) \rightarrow (\bigwedge_{i \in \mathcal{A}gt} \langle \preceq_i \rangle x))$$

Contrarily to *WPO*, *SPO* is a fairly complicated formula obtained directly from the definition and without much simplification. The next proposition states that these formalisations are correct.

Proposition 3. *Given an* HLCP *model* $\mathcal{M}$ *and a state* s *in* $\mathcal{M}$, s^* *is* weakly Pareto optimal *iff* $\mathcal{M}, s^* \models WPO$. *It is* strongly Pareto optimal *iff* $\mathcal{M}, s^* \models SPO$

PROOF. From Definition 15, for *WPO* we obtain $\downarrow x.\neg\langle\emptyset\rangle[\downarrow y.\bigwedge_{i \in \mathcal{A}gt}((@_x \langle \preceq_i \rangle y) \wedge \neg\langle \preceq_i \rangle x)]$. We simplify this by the observation that, from (total), $s \wedge \langle\emptyset\rangle t \wedge \neg\langle \preceq_i \rangle t \rightarrow @_t \langle \preceq_i \rangle s$ is a theorem of HLCP. *SPO* is straightforward from the definition with minor rewriting. ■

[2] Note that this is perfectly uniform with the weak case, since due to the reflexivity of $[\preceq_i]$ we have $WBR_i \leftrightarrow \downarrow x.[\overline{\{i\}}](\neg x \rightarrow \langle \preceq_i \rangle x)$.

We omit the proofs for the other equilibria. They all consist in translating the definitions of Section 4.2 and rewriting the formulation.

Dominance equilibria We define *very weak dominance*, *weak dominance* and *strict dominance*. Our definitions of dominance largely make use of the concept of best response.

A agent is currently playing a *very weakly dominant* strategy if this is its (weak) best response whatever what the other agents play. It should be clear now that we just have to formalise it via an agentive formula stating that "i chooses that it plays its best response whatever other agents do". Thus we characterise a state where i plays a very weakly dominant strategy by the formula $[i]WBR_i$. We then capture a very weak dominance equilibrium by:

$$VWSD =_{def} \bigwedge_{i \in \mathcal{A}gt} [i]WBR_i$$

Weak dominance imposes the strategy to be the strict best response to at least one of the possible combination of choice of the other agents, and this is the only difference with weak dominance. This is formalised by $\langle i \rangle SBR_i$. Thus, we characterise a state where i plays a weakly dominant strategy by the formula $[i]WBR_i \wedge \langle i \rangle SBR_i$, and we capture a weak dominance equilibrium by

$$WSD =_{def} \bigwedge_{i \in \mathcal{A}gt} [i]WBR_i \wedge \langle i \rangle SBR_i$$

Strict dominance is intuitively along the same line as very weak dominance, substituting the weak best response by the strict one (or the weak preference modality by a strict one). We characterise a strict dominance equilibrium by

$$SSD =_{def} \bigwedge_{i \in \mathcal{A}gt} [i]SBR_i$$

Proposition 4. *Given an* HLCP *model* $\mathcal{M}$ *and a state* s^* *in* $\mathcal{M}$, s^* *is*

1. very weakly dominant *iff* $\mathcal{M}, s^* \models VWSD$*;*
2. weakly dominant *iff* $\mathcal{M}, s^* \models WSD$*;*
3. strictly dominant *iff* $\mathcal{M}, s^* \models SSD$*.*

It is routine to check that strict strategy dominance implies weak strategy dominance which in turn implies very weak strategy dominance.

Proposition 5. $\vdash SSD \rightarrow WSD$ *and* $\vdash WSD \rightarrow VWSD$

Nash equilibria A state being a Nash equilibrium is simply defined by:

$$NE =_{def} \bigwedge_{i \in \mathcal{A}gt} WBR_i$$

A state is a Nash equilibrium if every agent uses its best response to the choice of the other agents. Remarkably, [25] proposed a similar definition along the pattern

$\bigwedge_i D_{\mathcal{A}gt\setminus\{i\}}\langle\preceq_i\rangle x$ within an epistemic language. (Recall our quick comparison in Section 3.2 between epistemic logic and our logic of choice.)

A state is a strong Nash equilibrium of the game if there is no coalition J that can change its choice and lead to a state considered strictly better by every members of J.

$$SNE =_{def} \downarrow x. \bigwedge_{J \subset \mathcal{A}gt} [\overline{J}](\bigvee_{i \in J} \langle \preceq_i \rangle x)$$

Proposition 6. *Given an* HLCP *model* $\mathcal{M}$ *and a state* s^* *in* $\mathcal{M}$, s^* *is a*

1. Nash equilibrium *iff* $\mathcal{M}, s^* \models NE$.
2. strong Nash equilibrium *iff* $\mathcal{M}, s^* \models SNE$.

The next proposition is straightforward.

Proposition 7. $\vdash SNE \rightarrow NE$

Core The use of HLCP is not restricted to non-cooperative games. We have already characterised strong Nash equilibrium. It is also easy to capture the concept of *core* of a cooperative strategic game without transferable payoff. We did not do so in our definition in Section 4.2, but as we did for Definition 8 we can start by giving the characterisation of an undominated state. A straightforward translation would be $DOM =_{def} \downarrow x.\langle\overline{J}\rangle \bigvee_{J \subset \mathcal{A}gt} [J] \bigwedge_{i \in J} \downarrow y.@_x\langle \prec_i \rangle y$.

$INCR$ is simply the negation of DOM. Up to equivalence (in particular because of (total), (agree), and modal distributivity/contraction) we obtain:

$$INCR =_{def} \downarrow x. \bigwedge_{J \subset \mathcal{A}gt} [\overline{J}]\langle J \rangle \bigvee_{i \in J} \langle \preceq_i \rangle x$$

Proposition 8. *Given an* HLCP *model* $\mathcal{M}$ *and a state* s^* *in* $\mathcal{M}$, s^* *is in the (weak)* core *iff* $\mathcal{M}, s^* \models INCR$.

Note the difference with (or the resemblance to) strong Nash equilibrium. We clearly have the following.

Proposition 9. $\vdash SNE \rightarrow INCR$

On the succinctness of solution concept characterisations As noted in the introduction, the number of strategies has no impact on the size of the characterisation of solution concepts in our logic. In the case of cooperative equilibria, the size of the characterisation depends on the number of coalitions, and is then exponential in the number of players. However, for all solution concepts but strong Nash equilibrium and core membership, the size of the formula is polynomial in the number of agents.

In summary, the syntax of HLCP allows to a designer to formalise important properties of games succinctly. This is a very desirable feature of a language when we are interested in model checking. There are at least two reasons for that: (i) less efforts are needed for the designer to write down a property to be tested, and (ii) the complexity of model checking is usually function of the size of the input formula.

5 Model checking

In order to verify properties of games, we can use the Hybrid Logic Model Checker (HLMC) [10]. This is an implementation of the algorithms of [12], where model checking of hybrid fragment including binders is proved PSPACE-complete when the size of the input formula is taken as parameter. (Model checking can be solved in polynomial time if the size of the model is the parameter.) HLMC is given a model and a formula. The output is the set of states in the model where the formula is satisfied, plus some statistics.

We present the model checking by means of two examples. This will allow us to demonstrate the ability of our logic with a wide assortment of properties. We first focus on solution concepts for which players are assumed to be individually rational. We define Nash equilibrium, very weak dominance and strict dominance in the language of HLMC (to be introduced). We also make explicit how an HLCP model is encoded. In a second part, we make a move to solution concepts for team reasoners: players are assume to be able to form coalitions. In the specification language of HLMC, we then define strong Nash equilibrium, core membership and the 'composite equilibrium' of Pareto optimal Nash equilibrium.

5.1 Equilibria of individual rationality

The language of HLMC for implementing the formulae to be tested matches with the logical representation. For example, we use `[ag1]` for $[\{1\}]$, `<pref2>` stands for $\langle \preceq_2 \rangle$, `B x` is the down-arrow binder $\downarrow x.$, `&` is the conjunction $\wedge$, `|` is the disjunction $\vee$, `!` is the negation $\neg$. We propose three progressive examples.

A Nash equilibrium in a 2-agent game is characterised by:

```
B x (
 ( [ag2]( <pref1>(x) ) )
 &
 ( [ag1]( <pref2>(x) ) )
)
```

A very weak dominant equilibrium in a 2-agent game is a slight modification of Nash equilibrium:

```
[ag1] ( B x ( [ag2]( <pref1>(x) ) ) )
&
[ag2] ( B x ( [ag1]( <pref2>(x) ) ) )
```

A strict dominant equilibrium in a 2-agent game is obtained from the very weak dominant equilibrium, expanding the definition of strict preferences:

```
[ag1] ( B x ( [ag2](!x ->
   ( B y (<pref1>( (x) & !<pref1>(y))) ) ) ))
&
[ag2] ( B x ( [ag1](!x ->
   ( B y (<pref2>( (x) & !<pref2>(y))) ) ) ))
```

The game of Figure 1 can be represented in the language of HLMC. It is the translation of the following definition of $\mathcal{M} = \langle \mathcal{A}gt, \mathcal{P}rop, \mathcal{N}om, \mathcal{WV}ar, S, (R_J), (P_i), \pi \rangle$ where:

– $\mathcal{A}gt = \{1, 2\}$;

- $\mathcal{P}rop = \emptyset$;
- $\mathcal{N}om = \{i_0, i_1, i_2, i_3\}$;
- $\mathcal{W}Var = \{x, y\}$;
- $S = \{s_0, s_1, s_2, s_3\}$;
- $R_\emptyset = \{(s, s') \mid s \in S, s' \in S\}$;
- $R_{\{1\}} = \{(s_0, s_1), (s_2, s_3)\}^*$, where $*$ is the equivalence closure;
- $R_{\{2\}} = \{(s_0, s_2), (s_1, s_3)\}^*$, where $*$ is the equivalence closure;
- $R_{\mathcal{A}gt} = \{(s, s) \mid s \in S\}$;
- $P_1 = \{(s_0, s_1), (s_2, s_0), (s_2, s_1), (s_2, s_3), (s_3, s_0), (s_3, s_1), (s_3, s_2)\}^*$, where $*$ is the reflexive closure;
- $P_2 = \{(s_0, s_3), (s_1, s_0), (s_1, s_2), (s_1, s_3), (s_3, s_0), (s_3, s_1), (s_3, s_2)\}^*$, where $*$ is the reflexive closure;
- $\pi(s_0) = \{i_0\}$, $\pi(s_1) = \{i_1\}$, $\pi(s_2) = \{i_2\}$, $\pi(s_3) = \{i_3\}$.

We give in Appendix the XML script which is the representation of this model. The following is a resume of the model generated by HLMC. Note that we did not give the relations of choice for the grand coalition and the empty coalition. The former is simply obtained as the identity relation, the latter is the composition of the relations of the two individual agents.

```
Kripke structure: XML
Worlds: s0 (0), s1 (1), s2 (2), s3 (3)
Modalities:
ag1 (0) = <s0, s0> <s0, s1> <s1, s0> <s1, s1> <s2, s2> <s2, s3>
        <s3, s2> <s3, s3>
ag2 (1) = <s0, s0> <s0, s2> <s1, s1> <s1, s3> <s2, s0> <s2, s2>
        <s3, s1> <s3, s3>
pref1 (2) = <s0, s0> <s0, s1> <s1, s1> <s2, s0> <s2, s1> <s2, s2>
        <s2, s3> <s3, s0> <s3, s1> <s3, s2> <s3, s3>
pref2 (3) = <s0, s0> <s0, s3> <s1, s0> <s1, s1> <s1, s2> <s1, s3>
        <s2, s2> <s3, s0> <s3, s1> <s3, s2> <s3, s3>
Propositional symbols:
Nominals:
i0 (0) = s0
i1 (1) = s1
i2 (2) = s2
i3 (3) = s3
```

We can now test some properties of this game. The result of the model checking in HLMC consists in giving the states satisfying the input formula and some statistics that we give such that the reader can have a grasp on the difference of resources needed for model checking the various properties.

Once the game encoded, we can verify that in all cases we obtain the expected output, that is, that the state s_0 corresponding to the action profile (a_1, a_2) is the only equilibrium of the three sorts tested. Figure 6 presents the results of model checking Nash equilibrium, weak strategy dominance and strict strategy dominance against the the previous model.

5.2 Equilibria for teams

It must be clear that the expressive power of HLCP is not limited the basic properties of games. The language is precise enough for specifying numbers of properties that one

formula	result	RT (in sec)	# recursive calls	# modal calls	# binder calls	max. nesting
NE	$\{s_0\}$	0.0000	45	16	1	7
WSD	$\{s_0\}$	0.0000	49	18	2	10
SSD	$\{s_0\}$	0.0000	273	74	10	17

Fig. 6. Experimental results on non-cooperative solution concepts.

a_3:

	a_2	b_2
a_1	$1, 0, -5$ (s_0)	$-5, -5, 0$ (s_1)
b_1	$-5, -5, 0$ (s_2)	$0, 0, 10$ (s_3)

b_3:

	a_2	b_2
a_1	$-1, -1, 5$ (s_4)	$5, -5, 0$ (s_5)
b_1	$-5, -5, 0$ (s_6)	$-2, -2, 0$ (s_7)

Fig. 7. A 3-player strategic game. Player 1 chooses rows, player 2 chooses columns and player 3 chooses matrices.

would like to verify. For instance, we can elaborate on equilibria that are desirable from the point of view of team reasoning.

On Figure 7, we have represented a strategic game involving three player. There are two Nash equilibria, (b_1, b_2, a_3) and (a_1, a_2, b_3), that are also strong Nash equilibria. Then, they are also in the core, which also contains (a_1, b_2, b_3). Perhaps a better solution of this game when players reason as a team is the concept of Pareto optimal Nash equilibrium. In this case (b_1, b_2, a_3) is the only solution.

We are now going to verify these statements with HLMC.

The internal representation of the corresponding model in HLMC is the following:

```
Kripke structure: XML
        Worlds: s0 (0), s1 (1), s2 (2), s3 (3), s4 (4), s5 (5), s6 (6), s7 (7)
        Modalities:
                ag1 (0) = <s0, s0> <s0, s1> <s0, s4> <s0, s5> <s1, s0> <s1, s1>
 <s1, s4> <s1, s5> <s2, s2> <s2, s3> <s2, s6> <s2, s7> <s3, s2> <s3, s3> <s3, s6>
 <s3, s7> <s4, s0> <s4, s1> <s4, s4> <s4, s5> <s5, s0> <s5, s1> <s5, s4> <s5, s5>
 <s6, s2> <s6, s3> <s6, s6> <s6, s7> <s7, s2> <s7, s3> <s7, s6> <s7, s7>
                ag2 (1) = <s0, s0> <s0, s2> <s0, s4> <s0, s6> <s1, s1> <s1, s3>
 <s1, s5> <s1, s6> <s1, s7> <s2, s0> <s2, s2> <s2, s4> <s3, s1> <s3, s3> <s3, s5>
 <s3, s7> <s4, s0> <s4, s2> <s4, s4> <s4, s6> <s5, s1> <s5, s3> <s5, s5> <s5, s7>
 <s6, s0> <s6, s2> <s6, s4> <s6, s6> <s7, s1> <s7, s3> <s7, s5> <s7, s7>
                ag3 (2) = <s0, s0> <s0, s1> <s0, s2> <s0, s3> <s1, s0> <s1, s1>
 <s1, s2> <s1, s3> <s2, s0> <s2, s1> <s2, s2> <s2, s3> <s3, s0> <s3, s1> <s3, s2>
 <s3, s3> <s4, s4> <s4, s5> <s4, s6> <s4, s7> <s5, s4> <s5, s5> <s5, s6> <s5, s7>
 <s6, s4> <s6, s5> <s6, s6> <s6, s7> <s7, s4> <s7, s5> <s7, s6> <s7, s7>
                ag12 (3) = <s0, s0> <s0, s4> <s1, s1> <s1, s5> <s2, s2> <s2, s6>
 <s3, s3> <s3, s7> <s4, s0> <s4, s4> <s5, s1> <s5, s5> <s6, s2> <s6, s6> <s7, s3>
 <s7, s7>
                ag13 (4) = <s0, s0> <s0, s1> <s1, s0> <s1, s1> <s2, s2> <s2, s3>
 <s3, s2> <s3, s3> <s4, s4> <s4, s5> <s5, s4> <s5, s5> <s6, s6> <s6, s7> <s7, s6>
 <s7, s7>
                ag23 (5) = <s0, s0> <s0, s2> <s1, s1> <s1, s3> <s2, s0> <s2, s2>
 <s3, s1> <s3, s3> <s4, s4> <s4, s6> <s5, s5> <s5, s7> <s6, s4> <s6, s6> <s7, s5>
 <s7, s7>
                pref1 (6) = <s0, s0> <s0, s5> <s1, s0> <s1, s1> <s1, s2> <s1, s3>
 <s1, s4> <s1, s5> <s1, s6> <s1, s7> <s2, s0> <s2, s1> <s2, s2> <s2, s3> <s2, s4>
 <s2, s5> <s2, s6> <s2, s7> <s3, s0> <s3, s3> <s3, s5> <s4, s0> <s4, s3> <s4, s4>
 <s4, s5> <s5, s5> <s6, s0> <s6, s1> <s6, s2> <s6, s3> <s6, s4> <s6, s5> <s6, s6>
 <s6, s7> <s7, s1> <s7, s3> <s7, s4> <s7, s5> <s7, s7>
                pref2 (7) = <s0, s0> <s0, s3> <s1, s0> <s1, s1> <s1, s2> <s1, s3>
```

```
<s1, s4> <s1, s5> <s1, s6> <s1, s7> <s2, s0> <s2, s1> <s2, s2> <s2, s3> <s2, s4>
<s2, s5> <s2, s6> <s2, s7> <s3, s0> <s3, s3> <s4, s0> <s4, s3> <s4, s4> <s5, s0>
<s5, s1> <s5, s2> <s5, s3> <s5, s4> <s5, s5> <s5, s6> <s5, s7> <s6, s0> <s6, s1>
<s6, s2> <s6, s3> <s6, s4> <s6, s5> <s6, s6> <s6, s7> <s7, s0> <s7, s3> <s7, s4>
<s7, s7>
               pref3 (8) = <s0, s0> <s0, s1> <s0, s2> <s0, s3> <s0, s4> <s0, s5>
<s0, s6> <s0, s7> <s1, s1> <s1, s2> <s1, s3> <s1, s4> <s1, s5> <s1, s6> <s1, s7>
<s2, s1> <s2, s2> <s2, s3> <s2, s4> <s2, s5> <s2, s6> <s2, s7> <s3, s3> <s4, s3>
<s4, s4> <s5, s1> <s5, s2> <s5, s3> <s5, s4> <s5, s5> <s5, s6> <s5, s7> <s6, s1>
<s6, s2> <s6, s3> <s6, s4> <s6, s5> <s6, s6> <s6, s7> <s7, s1> <s7, s2> <s7, s3>
<s7, s4> <s7, s5> <s7, s6> <s7, s7>
       Propositional symbols:
       Nominals:
               i0 (0) = s0
               i1 (1) = s1
               i2 (2) = s2
               i3 (3) = s3
               i4 (4) = s4
               i5 (5) = s5
               i6 (6) = s6
               i7 (7) = s7
```

We need to define the solution concepts that are relevant for this game. For three agents, Pareto optimal Nash equilibrium can be implemented in HLMC as follows:

```
B x (
  ( [ag23]( <pref1>(x) ) )
  &
  ( [ag13]( <pref2>(x) ) )
  &
  ( [ag12]( <pref3>(x) ) )
  &
  [ag12] ([ag3] ( <pref1>(x) | <pref2>(x) | <pref3>(x) ))
)
```

Observe that we did not use the global modality $[\emptyset]$ in the last clause (corresponding to Pareto optimality). As a consequence of $(elim_{[\emptyset]})$, it is indeed definable from two modalities $[J_1]$ and $[J_2]$ when $J_1 \cap J_2 = \emptyset$. Hence, we do not have to specify the relation of choice for the empty coalition in the input model.

Strong Nash equilibrium can be implemented as follows in HLMC:

```
B x (
  ( [ag1]( <pref2>(x) | <pref3>(x)) )
  &
  ( [ag2]( <pref1>(x) | <pref3>(x)) )
  &
  ( [ag3]( <pref1>(x) | <pref2>(x)) )
  &
  ( [ag23]( <pref1>(x)) )
  &
  ( [ag13]( <pref2>(x)) )
  &
  ( [ag12]( <pref3>(x)) )
)
```

Finally core membership can be implemented as follows:

```
B x (
  ( [ag1] (<ag23> ( <pref2>(x) | <pref3>(x)) ))
  &
  ( [ag2] (<ag13> ( <pref1>(x) | <pref3>(x)) ))
  &
  ( [ag3] (<ag12> ( <pref1>(x) | <pref2>(x)) ))
  &
```

```
  ( [ag23] (<ag1> ( <pref1>(x)) ))
  &
  ( [ag13] (<ag2> ( <pref2>(x)) ))
  &
  ( [ag12] (<ag3> ( <pref3>(x)) ))
)
```

Note that a solution concept defined for *k* agents can be used for model checking games of less than *k* players. All we shall need to do is to model the choices of the extra players as the vacuous and dummy choice. That is, every extra player will have not more power that the empty coalition.

We can now verify that our quick analysis of the solutions in the example is correct. Figure 8 presents the results of model checking Nash equilibrium, Pareto Optimal Nash equilibrium, strong Nash equilibrium and core membership against the the previous model.

formula	result	RT (in sec)	# recursive calls	# modal calls	# binder calls	max. nesting
NE	$\{s_3, s_4\}$	0.0000	137	48	1	8
NE ∧ *PO*	$\{s_3\}$	0.0000	289	88	1	14
SNE	$\{s_3, s_4\}$	0.0100	425	120	1	14
INCR	$\{s_3, s_4, s_5\}$	0.0200	473	168	1	15

Fig. 8. Experimental results on Nash equilibrium, weak Pareto Nash equilibrium and cooperative solution concepts.

6 Discussion and perspectives

We hope we have made clear that a logical language without action labels can be useful for model checking equilibrium in games. The main aspect is that when combined with the down arrow binder bringing the expressivity of "here and now" in the object language, it allows general characterisations of equilibria. With the small exception of [25], and as far as we know, such an approach has not been followed elsewhere.

Adding epistemic reasoning. A theory of interaction cannot be complete without epistemic attitudes. Since the action component of HLCP is inspired by STIT logics, a natural extension is to integrate knowledge, as in [9]. This simply consists of adding straightforward epistemic relations over states to the models and the underlying knowledge operators to the language. As a result we have an expressive logic capable of strategic reasoning under uncertainty.

As an illustration, the infamous notion of *knowing a strategy* is not ambiguous. (See [16] for an account of the problem in logics of ability.) We can distinguish: "*for all* epistemically indistinguishable states, *there exists* a strategy of J that leads to ϕ", from "*there exists* a strategy σ of the coalition J such that *for all* states epistemically indistinguishable for J, σ leads to ϕ". The former is a ∀-∃ schema of "knowing a strategy". It is in contrast to the latter sentence, which is a ∃-∀ schema.

The need for succinct models. It is not difficult to see that modelling even small strategic games is almost unfeasible. The HLMC basic constructor is:

```
<modality label="M">
                ...
<acc-pair to-world-label="s1" from-world-label="s0"/>
                ...
</modality>
```

stating that the relation underlying the modality `[M]` has an edge from the state `s0` to the state `s1`. Hence, given the language of HLMC the designer needs to specify *every* edge of every relation of the model.

Relations of choice. In the case of choice relations, every edge for reflexivity, transitivity and euclideanity must be specified. It is quite easy to see that we can encode choices efficiently. We could for instance use *ad hoc* constructors.

```
<choice-mod label="ag1">
            <equiv-class "s0 s1 s2">
</choice-mod>
```

would build all the edges to make $\{s_0, s_1, s_2\}$ an equivalence class representing a choice. Choice relations for coalitions can next be extrapolated from individual relations by intersection.

Relations of preferences. As the relations of preference are much less structured as the relations of choice, their case is also more problematic in practice. Given a game $\langle N, (A_i), (\succeq_i)\rangle$, a corresponding HLCP model $\langle \mathcal{A}gt, \mathcal{P}rop, \mathcal{N}om, \mathcal{W}Var, S, (R_J), (P_i), \pi\rangle$ will have $|S| = \prod_{i \in N} |A_i|$ states. Hence, only due to the totality of the preferences, for every agent i, $Card(P_i) \geq |S| + \frac{|S|(|S|-1)}{2}$. Then, for example, for any game of 3 players with 3 choices each, we need to specify at least 1134 edges of preference relation, and we still have to fix the transitivity!

From a practical point of view it means that HLMC is not optimal. It has to be associated with a piece of software taking a compact representation of the model in input and giving in output the XML script readable by HLMC. Such a 'black-box' can take inspiration from the research in compact representation of games. See for example [19, Sect. 2.5] for a short survey.

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Appendix: representation of the example in HLMC

We give the XML script which is the representation of the model pictured in Figure 1.

We first define four states representing the set of strategy profiles of the game. Then we enumerate explicitly every edge of the relations underlying the choices of agent 1, the choices of agent 2, the preferences of agent 1 and the preferences of agent 2. Finally we assign one nominal to each state. Remark that we did not give the relations of choice for the grand coalition and the empty coalition.


```
<?xml version="1.0" encoding="UTF-8"?>
<!DOCTYPE hl-kripke-struct SYSTEM "hl-ks.dtd">
<hl-kripke-struct name="XML">
  <world label="s0"/>
  <world label="s1"/>
  <world label="s2"/>
  <world label="s3"/>


  <!-- s0 is NE, VWSD and SSD
       (s0)1,1   (s1)2,0
       (s2)0,2   (s3)0,0 -->

  <modality label="ag1">
    <acc-pair to-world-label="s0" from-world-label="s0"/>
    <acc-pair to-world-label="s1" from-world-label="s1"/>
    <acc-pair to-world-label="s2" from-world-label="s2"/>
    <acc-pair to-world-label="s3" from-world-label="s3"/>

    <acc-pair to-world-label="s0" from-world-label="s1"/>
    <acc-pair to-world-label="s1" from-world-label="s0"/>

    <acc-pair to-world-label="s2" from-world-label="s3"/>
    <acc-pair to-world-label="s3" from-world-label="s2"/>
  </modality>

  <modality label="ag2">
    <acc-pair to-world-label="s0" from-world-label="s0"/>
    <acc-pair to-world-label="s1" from-world-label="s1"/>
    <acc-pair to-world-label="s2" from-world-label="s2"/>
    <acc-pair to-world-label="s3" from-world-label="s3"/>

    <acc-pair to-world-label="s0" from-world-label="s2"/>
    <acc-pair to-world-label="s2" from-world-label="s0"/>
    <acc-pair to-world-label="s1" from-world-label="s3"/>
    <acc-pair to-world-label="s3" from-world-label="s1"/>
  </modality>

  <modality label="pref1">
    <acc-pair to-world-label="s0" from-world-label="s0"/>
    <acc-pair to-world-label="s1" from-world-label="s1"/>
    <acc-pair to-world-label="s2" from-world-label="s2"/>
    <acc-pair to-world-label="s3" from-world-label="s3"/>

    <acc-pair to-world-label="s1" from-world-label="s0"/>
    <acc-pair to-world-label="s0" from-world-label="s2"/>
    <acc-pair to-world-label="s1" from-world-label="s2"/>
    <acc-pair to-world-label="s3" from-world-label="s2"/>
    <acc-pair to-world-label="s0" from-world-label="s3"/>
    <acc-pair to-world-label="s1" from-world-label="s3"/>
    <acc-pair to-world-label="s2" from-world-label="s3"/>
  </modality>


  <modality label="pref2">
    <acc-pair to-world-label="s0" from-world-label="s0"/>
    <acc-pair to-world-label="s1" from-world-label="s1"/>
    <acc-pair to-world-label="s2" from-world-label="s2"/>
    <acc-pair to-world-label="s3" from-world-label="s3"/>

    <acc-pair to-world-label="s3" from-world-label="s0"/>
    <acc-pair to-world-label="s0" from-world-label="s1"/>
    <acc-pair to-world-label="s2" from-world-label="s1"/>
    <acc-pair to-world-label="s3" from-world-label="s1"/>
    <acc-pair to-world-label="s0" from-world-label="s3"/>
    <acc-pair to-world-label="s1" from-world-label="s3"/>
    <acc-pair to-world-label="s2" from-world-label="s3"/>
  </modality>


  <nominal label="i0" truth-assignment="s0"/>
  <nominal label="i1" truth-assignment="s1"/>
  <nominal label="i2" truth-assignment="s2"/>
  <nominal label="i3" truth-assignment="s3"/>
</hl-kripke-struct>
```


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