A logic of propositional control for truthful implementations

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Abstract

We introduce a logic designed to support reasoning about social choice functions. The logic includes operators to capture strategic ability, and operators to capture agent preferences. We give a correspondence between formulae in the logic and properties of social choice functions, and show that the logic is expressively complete with respect to social choice functions, i.e., that every social choice function can be characterised as a formula of the logic. We show the decidability of the logic and give a complete axiomatization. To demonstrate the value of the logic, we show in particular how it can be applied to the problem of determining whether a social choice function is strategy-proof.

1 Introduction

Social choice theory (SCT) – the theory of collective decision-making in situations where preferences over the outcomes may differ – is a topic of fundamental importance in human society [Arrow et al., 2002]. For example, the design and analysis of voting procedures, such as those used in political elections across the world, has a direct effect on our lives.

Our aim in this work is to develop rigorous tools to assist in the analysis and design of social choice procedures. In particular, a long-term goal is to develop techniques that will permit the *automated* analysis of social choice procedures. To this end, we aim to develop languages that will allow us to formally express the properties of social choice procedures, such that these languages may be processed automatically and rigorously. Such languages can then be used as *query languages* for social choice procedures: given some property P of a social choice procedure (such as, e.g., the fact that the procedure is strategy-proof), we aim to be able to encode the property P as an expression ρ^P of our language, which we then pose as a query to an automated analysis system.

Our aim in the present paper is to set out a formal language intended for the specification of social choice properties. The language is basically that of a modal logic [Chellas, 1980], partially derived from the Coalition Logic of Propositional Control (CL-PC) [van der Hoek and Wooldridge, 2005]. The logic includes operators to capture strategic ability, and operators to capture agent preferences. After first recalling some key concepts from social choice and game theory, we introduce the logic. The basic idea is that we model an agent's preferences via atomic propositions: a proposition $p^i_{x>y}$ will be used to represent the fact that agent i has reported that he prefers outcome x at least as much as outcome y. The strategic abilities of agents are captured using a CL-PC-like operator: an agent can choose any assignment of values for its preference variables that corresponds to a preference ordering. After giving the syntax and semantics of the logic, we show how the logic can be used to characterise social choice functions, and show that the logic is expressively complete with respect to social choice functions, i.e., that every social choice function can be characterised as a formula of the logic. We give a complete axiomatization for the logic. To demonstrate the value of the logic, we formalise some properties of social choice functions and show in particular how it can be applied to the problem of determining whether a social choice function is strategy-proof.

2 Background

We present the basic definitions in game theory and social choice upon which we construct our framework. As the main references to the literature we use [Dasgupta et al., 1979] and [Osborne and Rubinstein, 1994].

We assume that game forms and social choice functions (to be defined hereafter) are over the same domains of agents and consequences. We denote by N the set of agents and by K the set of consequences. Typically, the agents are the voters and the consequences will be the candidates in some election. We denote by $L(K)$ the set of *linear* orders over K. (A linear order is a relation that is transitive, antisymmetric and total.) By using a linear order, we are assuming the players cannot be indifferent between two different alternatives. A relation of *preference* is a linear order. Given K and N, a *preference profile* $<$ is a tuple $(<_i)_{i\in N}$ of preferences, where $<_i \in L(K)$ for every i. The set of preference profiles is denoted by $L(K)^N$.

Definition 1 (social choice function) *Given K and N, a* social choice function *(SCF) is a single-valued mapping from the set $L(K)^N$ of preference profiles into the set K of outcomes.*

For every preference profile, a social choice function describes the desirable consequence (from the point of view of the designer).

Definition 2 (strategic game form) *A* strategic game form *is a tuple $\langle N, (A_i), K, o\rangle$ where:*

- *N is a finite nonempty set of* players *(or agents);*
- *A_i is a finite nonempty set of* actions *(or strategies) for each player $i \in N$;*
- *K is a finite nonempty set of outcomes;*
- *$o : \times_{i\in N} A_i \to K$ determines an outcome for every combination of actions.*

A strategic game form is sometimes called a *mechanism*. It specifies the agents taking part in the game, their available actions and what combinations of actions lead to. We refer to a collection $(a_i)_{i\in N}$, consisting of one action for every agent in N, as an *action profile*. Given an action profile a, we denote by a_i the action of the player i, and by a_{-i} the action profile of the coalition $N \setminus \{i\}$. For any $C \subseteq N$, we write a_C for the *coalitional actions* of those players that are a member of $A_C = \times_{j\in C} A_j$.

Remark 1 *One key observation it that there is a strong link between strategic game forms and social choice functions. Any social choice function can be viewed as a game form where the set of actions of every agent is $L(K)$, and the function o represents the choice function (see [Moulin, 1983]). For any SCF F, we denote the associated game form by g^F.*

A *strategic game* is basically the composition of a strategic game form with a preference profile, a collection of preference relations (one for every agent) over the set of consequences.

Definition 3 (strategic game) *A* strategic game *is a tuple $\langle N, (A_i), K, o, (<_i)\rangle$ where $\langle N, (A_i), K, o\rangle$ is a strategic game form, and for each player $i \in N$, $<_i$ is a* preference relation *over K.*

A solution concept defines for every game a set of action profiles, intuitively corresponding to action profiles that may be played through rational action. Exactly which solution concept is used depends upon the application at hand.

Definition 4 (solution concept) *A* solution concept *SC is a function that maps a strategic game form $\langle N, (A_i), K, o\rangle$ and a preference profile over K to a subset of the action profiles in A_N.*

Implementations.

We can now introduce the notion of implementation. The problem of implementation arises because a planner does not know the true preference profile of the players. Given a social choice function F involving a set of players N and a set of alternatives K, the planner only knows that every player $i \in N$ has a preference profile $<_i \in L(K)$. Assuming a pattern of behaviour – a solution concept SC – the role of the planner is then to design a mechanism (or game form) g such that for every possible preference profile $< \in L(K)^N$, the strategic game $\langle g, (<_i)\rangle$ admits at least one SC-equilibrium, and every SC-equilibrium leads to the consequence in K which is prescribed by the social choice function for the preference profile at hand, that is, the value of $F(<)$.

Definition 5 (implementation) *Given a solution concept SC, we say that the game form $g = \langle N, (A_i), K, o\rangle$ SC-*implements *the social choice function F if for every preference profile $< \in L(K)^N$ we have $o(a^*) = F(<)$ for every action profile $a^* \in SC(g, <)$.*

We say that the social choice function is *SC-implementable* if there is a game form that SC-implements it.

In some situations, an SCF can be implemented by a strategic game form whose space of action profiles corresponds to the space of preference profiles, and *telling the truth is an equilibrium*.

We call a strategic game form in which the set of strategies of a player i is the set of preferences over K a *direct mechanism*. Hence, each player is asked to report a preference, but not necessarily the true one. An appealing class of direct mechanisms are those in which reporting the true preference profile is an equilibrium. That is, the action profile where every player reports its true preference is an equilibrium of the game consisting of the direct mechanism and an arbitrary preference profile. We can define this notion for every solution concept SC.

Definition 6 (truthful implementation) *The direct mechanism $g = \langle N, (A_i), K, o\rangle$* truthfully SC-implements *the SCF F if for every preference profile $<$:*

1. *$a^* \in SC(g, <)$ where $a^*_i = <_i$ for every player i, and*
2. *$o(a^*) = F(<)$.*

We say that the social choice function is *truthfully SC-implementable* if there is a game form that truthfully *SC*-implements it. Note that truthful implementations only require that the report of the true preference profile is an equilibrium, but it is not required that this equilibrium is unique. In general, other equilibria could be present that would not lead to the outcome prescribed by the SCF. However, this notion of implementation can be motivated. Indeed, it is assumed that playing a direct mechanism, if casting the real preference is an equilibrium strategy, an agent would be sincere.

3 Logic of social choice functions

Following [Moulin, 1983], we propose here to model social choice functions as a particular kind of strategic game form. In [Troquard et al., 2009] we proposed a logic for modelling strategic games on the basis of CL-PC. Every player controls a set of propositional variables and a strategy for a player amounts to choosing a truth value for the variables he controls. We are going to reuse the ideas of our previous proposal and adapt them to game forms where the strategies of the players correspond to the reports of preferences.

Semantics.

We need to introduce some definitions and notation.

Let X be an arbitrary set of propositions. We can see a *valuation* of X as a subset $V \subseteq X$ where `tt` is assigned to the propositions in V and `ff` is assigned to the propositions in $X \setminus V$. We denote the set of possible valuations over X by Θ^X.

In presence of a set of players N and a set of consequences K, the set of propositions controlled by a player $i \in N$ is defined as $At[i, K] = \{p^i_{x>y} \mid x, y \in K\}$. Every $p^i_{x>y}$ is a proposition controlled by the agent i which means that i reports that it values the consequence x at least as good as y. We also define $At[N, K] = \cup_{i \in N} At[i, K]$, which is then the set of all controlled propositions.

We can 'encode' a particular preference (or linear order) of player i as a valuation of the propositions in $At[i, K]$. However, conversely, not all valuations correspond to a linear order preference. A strategy of a player i consists of reporting a valuation of $At[i, K]$ encoding a linear order over K. For every player i, we define *strategies*$[i, K]$ as a set of valuations $V \in \Theta^{At[i,K]}$ such that: (1) $p^i_{x>x} \in V$, (2) if $x \neq y$ then $p^i_{x>y} \in V$ iff $p^i_{y>x} \notin V$, and (3) if $p^i_{x>y} \in V$ and $p^i_{y>z} \in V$ then $p^i_{x>z} \in V$.

For every coalition $C \subseteq N$, we note *strategies*$[C, K]$ the set of tuples $v_C = (v_i)_{i \in C}$ where $v_i \in$ *strategies*$[i, K]$. It is the set of strategies of the coalition C. To put it another way, it corresponds to a valuation of the propositions controlled by the players in C, encoding one preference over K for every player in C.

A *state* (or reported preference profile) is an element of *strategies*$[N, K]$, that is, a strategy of the coalition containing all the players.

We now define the models of social choice functions.

Definition 7 (model of social choice functions) *A model of social choice functions over* N *and* K *is a tuple* $M = \langle N, K, out, (<_i)\rangle$*, such that:*

- $N = \{1, \ldots, n\}$ *is a finite nonempty set of players;*
- K *is a finite nonempty set of consequences;*
- out : *strategies*$[N, K] \longrightarrow K$ *maps every state to a consequence;*
- *For every* $i \in N$, $<_i \in L(K)$ *is the true preferences of* i*.*

Hence, every player i has two levels of preferences: (1) a true one, given in the model by $(<_i)$ and (2) a reported one, given by a valuation in *strategies*$[i, K]$.

Taking out the true preference profile from a model of SCF, we obtain a mere instantiation of pre-Boolean games [Bonzon et al., 2007]. It is required to assign every variable to one (actual control) and only one (exclusive control) player, but there are some constraints on the possible valuations ('non-full' control). In [Bonzon et al., 2007], actual and exclusive control are grasped by an assignment function (mapping every propositional variable to exactly one player), and the partial control is a consequence of a *set of constraints* given as a set of satisfiable propositional formulae.

The language $\mathcal{L}^{scf}[N, K]$ is inductively defined by the following grammar:

$$\varphi ::= \top \mid p \mid x \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond_C \varphi \mid \blacklozenge_i \varphi.$$

where p is atom of $At[N, K]$, x is an atom of K, $i \in N$, and C is a coalition. $\Diamond_C \varphi$ reads that provided that the players outside C hold on to their current strategy, the coalition C has a strategy for φ. $\blacklozenge_i \varphi$ reads that i locally (at the current state) prefers a reported profile where φ is true.

Definition 8 (truth values of $\mathcal{L}^{scf}[N, K]$) *Given a model* $M = \langle N, K, out, (<_i)\rangle$*, we are going to interpret formulas of* $\mathcal{L}^{scf}[N, K]$ *in a state of the model. A state* $v = (v_1, \ldots, v_n)$ *in* M *is a tuple of valuations* $v_i \in$ *strategies*$[i, K]$*, one for each agent. The truth definition is inductively given by:*

$M, v \models p$	*iff*	$p \in v_i$ *for some* $i \in N$
$M, v \models x$	*iff*	$out(v) = x$
$M, v \models \neg\varphi$	*iff*	$M, v \not\models \varphi$
$M, v \models \varphi \vee \psi$	*iff*	$M, v \models \varphi$ *or* $M, v \models \psi$
$M, v \models \Diamond_C \varphi$	*iff*	*there is a state* u *such that* $v_i = u_i$ *for every* $i \notin C$ *and* $M, u \models \varphi$
$M, v \models \blacklozenge_i \varphi$	*iff*	*there is a state* u *such that* $out(v) <_i out(u)$ *and* $M, u \models \varphi$

The truth of φ in all models over a set of players N and a set of consequences K is denoted by $\models_{\Lambda^{scf}[N,K]} \varphi$. The classical operators $\wedge$, $\rightarrow$, $\leftrightarrow$ can be defined as usual. We also define $\Box_C \varphi \triangleq \neg\Diamond_C \neg\varphi$ and $\blacksquare_i \varphi \triangleq \neg\blacklozenge_i \neg\varphi$.

Theorem 1 (decidability) *The problem of deciding whether a formula $\varphi \in \mathcal{L}^{scf}[N,K]$ is satisfiable is decidable.*

Proof. It suffices to remark that we can enumerate every model of SCF over N and K and check whether φ is satisfiable in one state of one model. ■

Ballots.

We will think of a particular preference of $L(K)$ encoded in the language of the propositions as a *ballot*.

Definition 9 (ballot) *For every player $i \in N$, we can see every $<_i \in L(K)$ as a permutation $[x_1, x_2 \ldots]$ of the elements of K, where the more to the left the consequence is, the more it is preferred by the player i. We can reify in the language the reported preferences, that is, the ballot casted by the player i:*

$$\mathsf{ballot}_i(<) \triangleq p^i_{x_1>x_2} \wedge p^i_{x_2>x_3} \wedge \ldots p^i_{x_{|K|-1}>x_{|K|}}.$$

Then, the formula

$$\mathsf{ballot}(<) \triangleq \bigwedge_{i\in N} \mathsf{ballot}_i(<)$$

is a reification of the reported preference profile $< = (<_1, \ldots, <_n)$, consisting of one ballot for every player $i \in N$.

Remark 2 *Note that for every $< \in L(K)$, the formula* ballot(<) *is true at one and only one state. The reader familiar with Hybrid Logic [Areces and ten Cate, 2006] may think of* ballot(<) *as a* nominal*, viz. a state label available in the object language.*

Example 1 *Suppose that $N = \{1, 2\}$ and $K = \{a, b, c\}$. Let a preference profile $(<^{ex}_1, <^{ex}_2) \in L(K)^N$ given by the data of the two permutations $[a, c, b]$ and $[c, a, b]$ representing respectively the preferences of player 1 and 2. This reported preference profile can be represented in the language $\mathcal{L}^{scf}[\{1, 2\}, \{a, b, c\}]$ by the formula*

$$\mathsf{ballot}(<^{ex}) \triangleq p^1_{a>c} \wedge p^1_{c>b} \wedge p^2_{c>a} \wedge p^2_{a>b}.$$

It is easy to verify that the constraints on the elements of strategies$[1, K]$ and strategies$[2, K]$ are sufficient for inferring a complete characterisation of the preference profile. The following is valid in the models of social choice functions over $\{1, 2\}$ and $\{a, b, c\}$:

$$\begin{aligned}\mathsf{ballot}(<^{ex}) \quad \leftrightarrow \quad & p^1_{a>a} \wedge p^1_{b>b} \wedge p^1_{c>c} \wedge p^1_{a>c} \wedge \\ & p^1_{c>b} \wedge p^1_{a>b} \wedge \neg p^1_{c>a} \wedge \neg p^1_{b>c} \wedge \\ & \neg p^1_{b>a} \wedge p^2_{a>a} \wedge p^2_{b>b} \wedge p^2_{c>c} \wedge \\ & p^2_{c>a} \wedge p^2_{a>b} \wedge p^2_{c>b} \wedge \neg p^2_{a>c} \wedge \\ & \neg p^2_{b>a} \wedge \neg p^2_{b>c}\end{aligned}$$

Characterising an SCF.

The logic of social choice functions provides a formal language that allows us to represent social choice functions syntactically.

Observe that $\Diamond_N$ plays the role of an existential modality: it allows us to quantify over all the possible valuations in $\Theta^{At[N,K]}$, or ballots.

Definition 10 (SCF characterisation) *We say that the formula $\rho^F \in \mathcal{L}^{scf}[N,K]$* characterises *the social choice function F if for all $< \in L(K)^N$ and $x \in K$ we have:*

$$F(<) = x \textit{ iff } \models_{\Lambda^{scf}[N,K]} \rho^F \rightarrow \Diamond_N(\mathsf{ballot}(<) \wedge x).$$

It is easy to see that the logic is expressively complete wrt. social choice functions. That is, for every SCF F over a set of players N and a set of consequence K, there exists a formula $\rho^F \in \mathcal{L}^{scf}[N,K]$ characterising it. Even though it may not be optimal in terms of succinctness, it suffices to consider the conjuncts of formulae $\Diamond_N(\mathsf{ballot}(<) \wedge x)$, for $< \in L(K)$ and $F(<) = x$. The next example shows, using a simple scenario, that we can obtain less naïve and more compact characterisations.

Example 2 *Consider the following model of SCF (or game form) where player 1 chooses rows, player 2 chooses columns and player 3 chooses matrices. There are two consequences a and b. Hence, every player i controls the set of atoms $\{p^i_{a>a}, p^i_{b>b}, p^i_{a>b}, p^i_{b>a}\}$. Every player i has two strategies: $p^i_{a>a} \wedge p^i_{b>b} \wedge p^i_{a>b} \wedge \neg p^i_{b>a}$ and $p^i_{a>a} \wedge p^i_{b>b} \wedge \neg p^i_{a>b} \wedge p^i_{b>a}$, that we denote respectively by $[a, b]$ and $[b, a]$. (In the logic $\Lambda^{scf}[\{1, 2, 3\}, \{a, b\}]$, they are in fact equivalent to the formulae $p^i_{a>b}$ and $p^i_{b>a}$, respectively.)*

$[a, b]$

	$[a, b]$	$[b, a]$
$[a, b]$	a	a
$[b, a]$	a	b

$[b, a]$

	$[a, b]$	$[b, a]$
$[a, b]$	a	b
$[b, a]$	b	b

We can represent it in the logic $\Lambda^{scf}[\{1, 2, 3\}, \{a, b\}]$ of social choice functions by the formula:

$$\rho^F \triangleq a \leftrightarrow (p^1_{a>b} \wedge p^2_{a>b}) \vee (p^1_{a>b} \wedge p^3_{a>b}) \vee (p^2_{a>b} \wedge p^3_{a>b}).$$

Note that since out is functional, in the models of social choice functions with $K = \{a, b\}$ the consequence b will hold whenever a does not.

True preferences.

From our basic language $\mathcal{L}^{scf}[N,K]$, we can also define an operator of interest concerning preferences. We can define the *global binary* operator of preferences $\psi \blacktriangleleft_i \varphi$, corresponding to a preference between propositions. It reads "all φ are better than all ψ".

$$\psi \blacktriangleleft_i \varphi \triangleq \Box_N \bigvee_{< \in L(K)^N} (\mathsf{ballot}(<) \wedge (\varphi \to \Box_N(\psi \to \blacklozenge_i \mathsf{ballot}(<)))).$$

The agent i prefers the proposition ψ over φ iff when the reported preference profile is $<$ and φ holds at the state labeled by $\mathsf{ballot}(<)$, then, whenever ψ holds in a state, i would prefer the state labeled by $\mathsf{ballot}(<)$ (cf. Remark 2).

Now, like in Definition 9 for reported preferences, we can now reify the true preferences. Provided that x and y are two possible consequences, the formula $y \blacktriangleleft_i x$ captures the fact the player i prefers (globally) the alternative y over the alternative x. Hence, from a preference profile $< \in L(K)^N$, we reify the preference $[x_1, x_2 \ldots]$ of the player i as follows:

$$\mathsf{true}_i(<) \triangleq (x_{|K|} \blacktriangleleft_i x_{|K|-1}) \wedge \ldots \wedge (x_3 \blacktriangleleft_i x_2) \wedge (x_2 \blacktriangleleft_i x_1).$$

Then, the formula

$$\mathsf{true}(<) \triangleq \bigwedge_{i \in N} \mathsf{true}_i(<)$$

is a reification of the true preference profile $<= (<_1, \ldots, <_n)$.

Remark 3 *Whenever in a model of social choice function M the true preference of a player i is such that $x <_i y$, then the formula $x \blacktriangleleft_i y$ is true at every state of M. However, the other way around does not hold. Indeed, when either x or y is not a possible consequence of a model, the formula $x \blacktriangleleft_i y$ is always true for every i. The object language does not allow to talk about true preferences on impossible consequences.*

Axiomatics.

The axiomatization of the models of social choice functions is presented in Figure 1.

Constraints of control (*refl*), (*antisym-total*) and (*trans*) say that every player casts an appropriate valuation of its controlled atoms: a valuation must encode a linear order. (*comp*∪) defines the local ability of coalitions in terms of local abilities of sub-coalitions. Transitivity of the operator $\Box_C$ is a consequence of (*comp*∪). Hence, together with $(T(i))$ and $(B(i))$, it makes of $\Box_C$ an $S5$ modality. (*empty*) means that the empty coalition has no power. (*exclu*) means that if an atom is controlled by a player i, the other players cannot change its value. (*ballot*) makes sure that an agent is always locally able to cast any preference. From (*comp-At*), provided that δ_1 and δ_2 do not contain a common controlled atom, if a coalition C_1 can locally enforce δ_1 and C_2 can locally enforce δ_2 then they can enforce $\delta_1 \wedge \delta_2$ together.

Axiom (*func*1) forces the fact that for every action profile there is one and only one outcome. (*func*2) ensures that outcomes are only determined by the valuations. (*incl*) ensures that if something is settled, a player cannot prefer its negation. $(4(\leq_i))$ characterises transitivity. (*antisym*′) and (*total*′) force that the relation of preference over states is antisymmetric and total. Finally, (*unifPref*) specifies a fundamental interaction between preferences and the outcomes. If the casted preference profile at hand leads to x and agent i prefers an action profile leading to y, then at every action profile leading to x, agent i will prefer every action profile leading to y, that is, all y are better than all x.

The logic has a clear flavour of normal modal logic [Chellas, 1980]. The presence of $(K(i))$ with the necessitation rule $(Nec(\Box_i))$ gives to the operator $\Box_i$ the property of normality. The necessitation rule for the operator $\blacksquare_i$ holds because of $(Nec(\Box_i))$ and the axioms (*comp*∪) and (*incl*). The normality of the modality $\blacksquare_i$ then follows from $(K(<_i))$.

The axiomatics is largely inspired by the axiomatics of the logic of games and propositional control (henceforth LGPC) presented in [Troquard et al., 2009]. The logic LGPC is designed to model strategic games in general. The agents have arbitrary strategies, and preferences allowing for indifference between two different consequences. On the other hand, in this paper we focus on SCFs and hence on particular strategic games that 'represent' an SCF (cf. Remark 1).

While in LGPC we had an axiom saying that every atom was actually controlled by at least one agent, here we are more specific as we know *a priori* which atoms are controlled by a given agent. This is the role of the axiom (*ballot*). Constraints of controls are also specific to the present study. The truth values of the controlled atoms cannot be independent of each other as we use them to encode preferences. In LGPC, all valuations of the controlled atoms were permitted.

Theorem 2 (soundness and completeness) $\Lambda^{scf}[N, K]$ *is sound and complete with respect to the class of models of social choice functions.*

Proof. The proof of completeness first gives an equivalent but more standard semantics, based on Kripke models, to the logic (Kripke models of SCF). Then we build the canonical model. For every consistent formula φ, we show how to isolate a sub-model M_φ that we prove is a Kripke model of SCF that satisfies φ.

Further details are given in the Appendix. ■

4 Applications

We have already demonstrated that the language allows to completely characterise an SCF. In this section we show

Constraints of control		
(refl)	$p^i_{x>x}$	
$(\mathit{antisym\text{-}total})$	$p^i_{x>y} \leftrightarrow \neg p^i_{y>x}$	, where $x \neq y$
(trans)	$p^i_{x>y} \wedge p^i_{y>z} \rightarrow p^i_{x>z}$	
Propositional control		
(Prop)	φ	, where φ is a propositional tautology
$(K(i))$	$\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$	
$(T(i))$	$\Box_i\varphi \rightarrow \varphi$	
$(B(i))$	$\varphi \rightarrow \Box_i\Diamond_i\varphi$	
$(\mathit{comp}\cup)$	$\Box_{C_1}\Box_{C_2}\varphi \leftrightarrow \Box_{C_1\cup C_2}\varphi$	
(empty)	$\Box_{\emptyset}\varphi \leftrightarrow \varphi$	
(exclu)	$(\Diamond_i p \wedge \Diamond_i\neg p) \rightarrow (\Box_j p \vee \Box_j \neg p)$	, where $j \neq i$
(ballot)	$\Diamond_i \mathsf{ballot}_i(<)$	
$(\mathit{comp\text{-}At})$	$\Diamond_{C_1}\delta_1 \wedge \Diamond_{C_2}\delta_2 \rightarrow \Diamond_{C_1\cup C_2}(\delta_1 \wedge \delta_2)$	
Consequences and preferences		
$(\mathit{func}1)$	$\bigvee_{x\in K}(x \wedge \bigwedge_{y\in K\setminus\{x\}} \neg y)$	
$(\mathit{func}2)$	$(\mathsf{ballot}(<) \wedge \varphi) \rightarrow \Box_N(\mathsf{ballot}(<) \rightarrow \varphi)$	
(incl)	$\Box_N\varphi \rightarrow \blacksquare_i\varphi$	
$(K(<_i))$	$\blacksquare_i(\varphi \rightarrow \psi) \rightarrow (\blacksquare_i\varphi \rightarrow \blacksquare_i\psi)$	
$(4(<_i))$	$\blacklozenge_i\blacklozenge_i\varphi \rightarrow \blacklozenge_i\varphi$	
$(\mathit{antisym}')$	$(\mathsf{ballot}(<) \wedge \blacklozenge_i\mathsf{ballot}(<')) \rightarrow \Box_N(\mathsf{ballot}(<') \rightarrow \blacksquare_i\neg\mathsf{ballot}(<))$	
(total')	$(\mathsf{ballot}(<) \wedge \blacklozenge_i\mathsf{ballot}(<')) \vee \Box_N(\mathsf{ballot}(<') \rightarrow \blacklozenge_i\mathsf{ballot}(<))$	
$(\mathit{unifPref})$	$(x \wedge \blacklozenge_i y) \rightarrow (x \blacktriangleleft_i y)$	
Rules		
(MP)	from $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ infer $\vdash \psi$	
$(\mathit{Nec}(\Box_i))$	from $\vdash \varphi$ infer $\vdash \Box_i\varphi$	

Figure 1: *Logic of social choice functions* $\Lambda^{scf}[N, K]$. i ranges over N, C_1 and C_2 over 2^N, x and y are over K, and $<$ is over $L(K)^N$. δ_1 and δ_2 are two formulae from $\mathcal{L}^{scf}[N, K]$ that do not contain a common atom from $At[N, K]$. φ represents an arbitrary formula of $\mathcal{L}^{scf}[N, K]$, and p an arbitrary atom in $At[N, K]$.

how we can express properties of social choice functions in the language and apply the logic to reason about them.

The language can be used to characterise requirements on social choice functions. We first illustrate that on simple properties, namely citizen sovereignty and non-dictatorship. Next, we will characterise a dominant strategy equilibrium. Finally, we provide a formalisation of monotonicity and strategy-proofness, and use standard results of SCT to show how we can use the logic to check whether an SCF is implementable in dominant strategy.

Citizen sovereignty and non dictatorship.

We say that an SCF satisfies *citizen sovereignty* iff every consequences in K is possible. That is, no consequence is rejected independently of the individual opinions. It is defined as follows.

Definition 11 (citizen sovereignty) *An SCF F satisfies* citizen sovereignty *iff for every $x \in K$ there is a $< \in L(K)^N$ such that $F(<) = x$.*

The next formula is a straightforward translation of the definition of citizen sovereignty in the language of social choice functions.

$$\mathsf{CITSOV} \triangleq \bigwedge_{x\in K} \Diamond_N x.$$

We say that an SCF satisfies *non dictatorship* iff no player can always impose its favorite consequence.

Definition 12 (non dictatorship) *An SCF F is* non dictatorial *iff for every player $i \in N$ there is a ballot $< \in L(K)^N$ such that $F(<) <_i y$ for some $y \in K \setminus \{F(<)\}$.*

It says that for very player, there is a ballot $<$ whose consequence is $F(<)$, and i values better a consequence that is not $F(<)$.

We can rewrite the definition of non dictatorship into the language of social choice functions as follows.

$$\mathsf{NODICT} \triangleq \bigwedge_{i\in N} \Diamond_N \left(\bigvee_{x\in K} \left(x \wedge \bigvee_{y\in K\setminus\{x\}} p^i_{y>x} \right) \right).$$

The following proposition is immediate.

Proposition 1 *Consider a social choice function F and ρ^F a formula characterising it.*

- *F has the property of citizen sovereignty iff* $\models_{\Lambda^{scf}[N,K]} \rho^F \to \mathsf{CITSOV}$.
- *F is non dictatorial iff* $\models_{\Lambda^{scf}[N,K]} \rho^F \to \mathsf{NODICT}$.

Dominant strategy equilibrium.

Citizen sovereignty and non dictatorship are properties of social choice function: their formulations in logic are globally true (or false) in a model of SCF. However, the logic is also able to formalise solution concepts, which are properties of states. In [Troquard et al., 2009], we have characterised several solution concepts (dominant strategy equilibrium, Nash equilibrium, core membership...) that are directly applicable in the logic of the present work.

In order to formalise strategy-proofness later, we need to characterise a dominant strategy equilibrium. A dominant strategy equilibrium captures a particularly important pattern of behaviour. It arises when every player plays a dominant strategy, that is, a strategy that would be the best deviation whatever the other agents play. We define it directly in our models of SCF.

Definition 13 (dominant strategy equilibrium) *Let v^* be a state in a model of social choice functions $\langle N, K, out, (<_i)\rangle$. v^* is a* dominant strategy equilibrium *iff for every player $i \in N$ and every strategy $u_{N\setminus\{i\}} \in strategies[N \setminus \{i\}, K]$, we have $out((u_0 \dots u'_i \dots u_n)) <_i out((u_0 \dots v^*_i \dots u_n))$ for every $u'_i \in strategies[i, K]$.*

A dominant strategy equilibrium is a strong solution concept: such an equilibrium does not depend on the knowledge of an agent i about the strategies or preferences of other players.

It is convenient to introduce the notion of best response by an agent i.

$$\mathsf{BR}_i \triangleq \bigvee_{x\in K}(x \land \Box_i \blacklozenge_i x).$$

A player i plays a best response in a state if, x being the outcome, for every deviation of i, i prefers x.

We can now define strategy dominance in terms of best response:

$$\mathsf{DOM} \triangleq \bigwedge_{i\in N} \Box_{N\setminus\{i\}}\mathsf{BR}_i.$$

We have a strategy dominant state if the current choice of every player ensures them a best response whatever other agents do.

Proposition 2 *Assume a model of social choice functions M and a state v^*. We have that v^* is a dominant strategy equilibrium iff $M, v^* \models \mathsf{DOM}$.*

Monotonicity and strategy-proofness.

One important property of SCF is *monotonicity* as it can bear on the implementability of social choice functions.

Definition 14 (monotonicity) *An SCF F is* monotonic *iff for all $\{<, <'\} \subseteq L(K)^N$ and $x \in K$, if $F(<) = x$ and if for all $i \in N$, for all $y \in K$ we have that that $y <_i x$ implies that $y <'_i x$, then, $F(<') = x$.*

We propose to characterise monotonic social choice functions. We define

$$\mathsf{MON} \triangleq \left\{ \begin{array}{c} \bigwedge_{<\ \in L(K)^N} \bigwedge_{<'\ \in L(K)^N} \bigwedge_{x\in K}\Big[\Diamond_N(\mathsf{ballot}(<) \land x)\land \\ \bigwedge_{i\in N} \bigwedge_{y\in K}\Big(\Diamond_N(\mathsf{ballot}(<) \land p^i_{x>y}) \to \\ \Diamond_N(\mathsf{ballot}(<') \land p^i_{x>y})\Big) \to \Diamond_N(\mathsf{ballot}(<') \land x)\Big] \end{array} \right\}.$$

Again, the predicate MON is merely a rewriting of Definition 14 in our language of social choice functions $\mathcal{L}^{scf}[N, K]$. The following proposition is immediate.

Proposition 3 *Consider a social choice function F and ρ^F a formula characterising it. F is monotonic iff*

$$\models_{\Lambda^{scf}[N,K]} \rho^F \to \mathsf{MON}.$$

Monotonicity does not depend on the true preference profile of the players. Accordingly, our definition does not involve the modalities of preference $\blacklozenge_i\varphi$ and $\varphi \blacktriangleleft_i \psi$. Capitalising on standard results from social choice theory, we will show that using the full expressivity of our language (that is, using true preference modalities) we can obtain a much simpler formulation.

We say that an SCF is strategy-proof if for every preference profile, telling the truth (revealing the true preference) is a dominant strategy for every player.

Definition 15 (strategy-proofness) *An SCF F is* strategy-proof *iff F is truthfully* DOM*-implementable.*

Hence, a choice function is strategy-proof when it is truthfully implementable in dominant strategy: for every preference profile, reporting their true preference is a dominant strategy for every player.

The famous lemma called the *revelation principle* [Gibbard, 1973] is a central result in implementation theory. It states that if an SCF is DOM-implementable, then it is truthfully DOM-implementable. It is true in general even if $L(K)$ was containing non-strong orderings.

It means that if an SCF F is implementable in dominant strategies there exists a direct mechanism such that for every preference profile $<$, truth telling (every player i reports $<_i$) is a dominant strategy and the outcome is $F(<)$.

Truthful implementations are rather weak; it is easier in general to implement a choice function truthfully than with 'standard' implementations. Indeed, in truthful implementations there might be an equilibrium that leads to a consequence different of the one prescribed by the SCF. But because in this paper we consider linear preferences, and then that players cannot be indifferent between two consequences, such a situation cannot happen. Thus, we can be more specific than the revelation principle.

Theorem 3 ([Dasgupta et al., 1979, Corollary 4.1.4]) *A direct mechanism g truthfully implements an SCF F in dominant strategies iff g* DOM*-implements F.*

Hence, when working in dominant strategies with linear preferences, the concepts of implementation and truthful implementation coincide.

We propose to characterise strategy-proof social choice functions. We define

$$\mathsf{STRPROOF} \triangleq \bigwedge_{< \in L(K)^N} [\mathsf{true}(<) \to (\mathsf{ballot}(<) \to \mathsf{DOM})]$$

The formula STRPROOF is an immediate reformulation of the definition of strategy-proofness in our language of social choice functions.

Proposition 4 *Consider a social choice function F and ρ^F a formula characterising it. F is strategy-proof iff*

$$\models_{\Lambda^{scf}[N,K]} \rho^F \to \mathsf{STRPROOF}.$$

It provides us a general procedure to check whether a social choice function is strategy-proof. Moreover (because of Theorem 3), because we restricted our attention to linear preferences, it allows us to check whether an SCF is DOM-implementable.

Example 3 *We can verify that the social choice function characterised in Example 2 is strategy-proof.*

$\models_{\Lambda^{scf}[\{1,2,3\},\{a,b\}]} (a \leftrightarrow (p^1_{a>b} \wedge p^2_{a>b}) \vee (p^1_{a>b} \wedge p^3_{a>b}) \vee (p^2_{a>b} \wedge p^3_{a>b})) \to \mathsf{STRPROOF}.$

Monotonicity can bear implementability and this is actually the case in our setting. Since we are working with *rich domains* of preferences[1] and linear orderings the following result holds.

Theorem 4 ([Dasgupta et al., 1979, Cor. 3.2.3, Th. 4.3.1]) *An SCF is truthfully implementable in dominant strategies iff it is monotonic.*

Then, in our setting the notions monotonicity and strategy-proofness collapse together. Trivially we are actually able to substantially simplify MON, our definition of monotonicity in the formal language. Indeed, as a consequence of Theorem 4, we have the following.

Proposition 5

$$\models_{\Lambda^{scf}[N,K]} \mathsf{MON} \leftrightarrow \mathsf{STRPROOF}.$$

5 Discussion and perspectives

We have presented the problem of direct implementation in social choice theory and proposed a logical formalisation of it. We were able to give a sound and complete axiomatization.

We showed how we can characterise social choice functions and properties of social choice functions. Finally, we have demonstrated the value of the logic by proposing a general logical procedure for checking whether a social choice function is strategy-proof.

It is worth noting that the formalisation of the properties of SCF that we considered are almost immediate. Our logical language is a natural counterpart of the language of mathematics used in social choice theory. There are however two features that make it particularly useful: (1) it is supported by a non ambiguous semantics and (2) the resulting logic is decidable.

Section 3 suggests a logical methodology for reasoning about problems of social choice theory with the logic of social choice functions. Let a collection of properties of social choice theory $Pi, i \in \{1, \dots n\}$ characterised in the logic $\Lambda^{scf}[N, K]$ respectively by ρ^{Pi}.

1. *We can use the logic in order to check whether an SCF satisfies a certain property.* An SCF F characterised by ρ^F has the property $P1$ iff $\rho^F \to \rho^{P1}$ is derivable in $\Lambda^{scf}[N, K]$.
2. *We can use the logic in order to evaluate the strength of constraints in SCT.* $P1$ is a property weaker than $P2$ iff the formula $\rho^{P2} \to \rho^{P1}$ is derivable in $\Lambda^{scf}[N, K]$. For instance, instead of using a result of SCT to prove Proposition 5, we could actually use the logic to automatically verify that monotonicity and strategy-proofness coincide in the current setting. More interestingly, we could use it to prove new theorems.
3. *We can use the logic for mechanism design.* Building a mechanism that implements a social choice procedure satisfying the properties $P1, P2, \dots Pn$ consists of finding a model for the formula $\rho^{P1} \wedge \rho^{P2} \wedge \dots \wedge \rho^{Pn}$.

These are exciting perspectives and they are theoretically possible as the logic is decidable. Addressing the complexity of the problem of satisfiability and hopefully elaborating efficient proof methods for the logic is now needed.

[1]The notion of rich domain is tangential in this paper. Our domain of preferences is rich because we allow every linear orderings of K. See [Dasgupta et al., 1979, Sec. 3.1]

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Proof of Theorem 2

$\Lambda^{scf}[N, K]$ is sound and complete with respect to the class of models of social choice functions.

PROOF. *It is routine to verify that all principles of Figure 1 are valid. We show that if a formula is consistent, it is provable in the system* $\Lambda^{scf}[N, K]$.

We first introduce the Kripke models of SCF. A Kripke model of SCF *is a tuple* $M = \langle N, K, S, (R_i), (P_i), V\rangle$ *such that:*

- *N and K are parameters;*
- $S = \{V \in \Theta^{At[N,K]} \mid \forall i \in N, \exists V_i \in$ *strategies*$[i, K]$ *s.t.* $V = \cup_{i\in N} V_i\}$*;*
- *V is a valuation function of* $At[N, K] \cup K$ *where for every* $v \in S$*:*
  - $p \in V(v)$ *iff* $p \in v$, $p \in At[N, K]$*;*
  - *there is a unique* $x \in K$ *s.t.* $x \in V(v)$*;* [$\hookrightarrow$ we say that the model is *based* on the outcome function out^M when $out^M(v) = x$ iff $x \in V(v)$].
- $R_i vu$ *iff* $v_j = u_j$ *for all* $j \neq i$*;*
- *there is a* $<^M \in L(K)^N$ *s.t.* $P_i vu$ *iff (if* $x \in V(v)$ *and* $y \in V(u)$ *then* $x <^M_i y$*);* [$\hookrightarrow$ we say that the model is *based* on $<^M$].

Truth values of $\Diamond_i\varphi$ *and* $\blacklozenge_i\varphi$ *in a Kripke model of SCF are obtained in the standard way from the relations* R_i *and* P_i*, respectively.*

Clearly, for every Kripke model M based on out^M *and* $<^M$*, we can construct a model of social choice functions* $M^{scf} = \langle N, K, out^M, (<^M_i)\rangle$ *and reciprocally.*

By construction, there exists a bijection $f : S \longrightarrow$ *strategies*$[N, K]$ *that associates a state s in M to a state* $v = (v_1 \dots v_n)$ *in* M^{scf} *in such a way that for every* $p \in At[i, K]$*, we have* $p \in V(s)$ *iff* $p \in v_i$.

The following is easy to see.

Claim 1 $M, s \models \varphi$ *iff* $M^{scf}, f(s) \models \varphi$.

Hence, the proof of the theorem can be reduced to a proof of completeness of the logic wrt. to the class of Kripke models of SCF.

Let Ξ *be the set of maximally consistent sets (mcs.) of* $\Lambda^{scf}[N, K]$*. We define the proper canonical model* $M^{can} = \langle N, K, S, (R_i), (P_i), V\rangle$ *as follows. N and K are the parameters of the logic.* $S = \Xi$. $R_i\Gamma\Delta$ *iff* $\forall\delta \in \Delta$, $\Diamond_i\delta \in \Gamma$. $P_i\Gamma\Delta$ *iff* $\forall\delta \in \Delta$, $\blacklozenge_i\delta \in \Gamma$. $p \in V(\Gamma)$ *iff* $p \in \Delta$. $x \in V(\Gamma)$ *iff* $x \in \Delta$.

Given an mcs. Γ_0 *we define the set of mcs. 'describing' the same SCF* and *where the players have the same true preferences (modulo the preferences concerning some consequence which is not an outcome of the SCF):*

$$Cluster(\Gamma_0) \triangleq \{\Gamma_1 \mid \forall < \in L(K)^N, \forall x \in K, \Diamond_N(\mathsf{ballot}(<) \wedge x) \in \Gamma_1 \textit{ iff } \Diamond_N(\mathsf{ballot}(<) \wedge x) \in \Gamma_0\} \cap \{\Gamma_2 \mid \forall i \in N, \forall\{x, y\} \subseteq K, x \blacktriangleleft_i y \in \Gamma_2 \textit{ iff } x \blacktriangleleft_i y \in \Gamma_0\}$$

Let φ *be a consistent formula of* $\mathcal{L}^{scf}[N, K]$*. There is an mcs.* Γ_φ *s.t.* $\varphi \in \Gamma_\varphi$*. The proof consists in constructing a* *model from Γ_φ such that it is indeed a Kripke model of SCF and there is a state satisfying φ.*

We define $M_\varphi = \langle N', K', S', R'_i, P'_i, V'\rangle$ from M^{can} as follows:

- $N' = N$ *and* $K' = K$;
- $S' = \Xi_{|Cluster(\Gamma_\varphi)}$;
- $R'_i = R_{i|Cluster(\Gamma_\varphi)}$;
- $P'_i = P_{i|Cluster(\Gamma_\varphi)}$;
- $p \in V'(\Delta)$ *iff* $p \in V(\Delta)$, $\Delta \in S'$.

It is immediate that the truth lemma holds.

Claim 2 *$M_\varphi, \Gamma \models \delta$ iff $\delta \in \Gamma$.*

Hence, $M_\varphi, \Gamma_\varphi \models \varphi$.

The set of states in Kripke models of SCF is defined as the set of valuations of $At[N, K]$ encoding a preference profile. We prove that there exists a bijection between S' and $L(K)^N$.

Claim 3 *The following hold true:*

1. $\forall\Delta \in S', \exists! < \in L(K)^N$ *s.t.* $\mathsf{ballot}(<) \in \Delta$;
2. $\forall < \in L(K)^N, \exists!\Delta \in S'$ *s.t.* $\mathsf{ballot}(<) \in \Delta$.

The first part of the claim follows from the constraints of control (refl), (antisym-total) and (trans). We now argue that for every $< \in L(K)^N$, there is exactly one $\Delta \in S'$ such that $\mathsf{ballot}(<) \in \Delta$. Let $< \in L(K)^N$. We have $\vdash \Diamond_i \mathsf{ballot}_i(<)$ by (ballot). With (comp-At), we find that $\vdash \Diamond_N \mathsf{ballot}(<)$. Hence, $\Diamond_N \mathsf{ballot}(<) \in \Gamma_\varphi$, and there must be an mcs. Δ s.t. $\mathsf{ballot}(<) \in \Delta$. Now suppose that $\Delta' \in S'$ also contains $\mathsf{ballot}(<)$. By (func2), Δ and Δ' contain the same formulae. Then $\Delta' = \Delta$, which prove the second part of the claim.

As a consequence we will be allowed to use the formulae of the form $\mathsf{ballot}(<)$ as world labels in M_φ.

We now prove the main claim of this proof.

Claim 4 *M_φ is a Kripke model of SCF.*

We prove that for every mcs. Γ and Δ, we have that $R_i\Gamma\Delta$ iff for all $i \neq j$ ($\mathsf{ballot}_j(<) \in \Gamma$ iff $\mathsf{ballot}_j(<) \in \Delta$).

First, observe that for every i, R_i is an equivalence relation because by axioms ($K(i)$), ($T(i)$), ($B(i)$) and (comp$\cup$) all $\Box_i$ are S5 modalities.

($\Rightarrow$). Suppose $R_i\Gamma\Delta$. Then by definition $\forall\delta \in \Delta$ we have $\Diamond_i\delta \in \Gamma$. For any $< \in L(K)^N$ and $j \neq i$, suppose also that $\mathsf{ballot}_j(<) \in \Delta$. By (exclu), $\Box_i \mathsf{ballot}_j(<) \in \Delta$. Then by hypothesis $\Diamond_i\Box_i \mathsf{ballot}_j(<) \in \Gamma$, which by ($B(i)$) entails that $\mathsf{ballot}_j(<) \in \Gamma$. Because $R_i\Gamma\Delta$ is an equivalence relation, the same reasoning can be done to prove that if $\mathsf{ballot}_j(<) \in \Gamma$ then $\mathsf{ballot}_j(<) \in \Delta$.

($\Leftarrow$). Suppose $\forall j \neq i$, $\forall < \in L(K)^N$ we have $\mathsf{ballot}_j(<) \in \Gamma$ iff $\mathsf{ballot}_j(<) \in \Delta$.

Suppose that $\mathsf{ballot}_i(<') \in \Delta$ and $\delta \in \Delta$. Let us note $<_\Delta$ the preference profile $(<_1, \dots <'_i \dots <_n)$. We hence have $\mathsf{ballot}(<_\Delta) \wedge \delta \in \Delta$. Which by (func2) means that $\Box_N(\mathsf{ballot}(<) \rightarrow \delta) \in \Delta$.

From (exclu), $\Box_i \bigwedge_{j\neq i} \mathsf{ballot}_j(<) \in \Gamma$. By (ballot), we also have that $\Diamond_i \mathsf{ballot}_i(<') \in \Gamma$. Hence, by S5, $\Diamond_i \mathsf{ballot}(<_\Delta) \in \Gamma$.

We obtain that $\Diamond_i\delta \in \Gamma$.

We prove that there is a linear order $< \in L(K)^N$ such that $P_i\Gamma\Delta$ iff (if $x \in V(\Gamma)$ and $y \in V(\Delta)$ then $x <_i y$). For every $i \in N$, we construct an order $<^\circ_i$ over the set $K^\circ = \{x \in K \mid \Diamond_N x \in \Gamma_\varphi\}$ such that $x <^\circ_i y$ iff $x \blacktriangleleft_i y \in \Gamma_\varphi$.

Capitalising on (unifPref), it is immediate that $x <^\circ_i y$ is transitive ($4(<_i)$), antisymmetric (antisym$'$) and total and reflexive (total$'$). Then $<^\circ_i$ is a linear order over K°.

It is now easy to obtain a linear order $<_i$ over K such that for all x and y in K° we have $x <_i y$ iff $x <^\circ_i y$.

This completes the proof that M_φ is a Kripke model of SCF.

Then, for every consistent formula φ, there is a Kripke model of SCF in which φ is satisfied. ■